THE STABILITY OF CODIMENSION ONE BIFURCATIONS OF THE PLANAR REPLICATOR EQUATIONS

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To my Parents

CONTENTS

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Introduction

The theory of bifurcations of dynamical systems, or in other words the study of the topological metamorphoses of the phase portrait of a parameter dependent dynamical system as the parameter changes, has become a major field of research in pure mathematics and has been increasingly applied to all branches of physical and social sciences.

The theoretical groundwork of bifurcation theory, which is based on the work of Poincaré, Andronov and his school, and finally Thom, has been well formulated by Arnold [4] [5]. As long as we are studying a single dynamical system, non generic or degenerate systems which by arbitrary small perturbations are turned into generic or stable systems can be ignored. But when a whole family of dynamical systems is considered these non generic cases may be unremovable by the perturbations of the family in question. The simplest kind of degeneracies are unremovable in generic one parameter families; they are accordingly called codimension one degeneracies. Codimension k degeneracies are those which remain unremovable by generic k-parameter families. A complete study of a degenerate system always amounts to finding its codimension and investigating the bifurcation in the family for which the degeneracy is unavoidable. As a rule, the higher the codimension of a degenerate system the rarer is the degeneracy and the more difficult and in general the less useful is its study.

One then hopes to classify the systems at least in lower codimensions. The program is as follows. Given a degenerate system one looks at the deformations of that system i.e. local families or local unfoldings which contain that system. A "sufficiently large" deformation which "represents" all perturbations of the degeneracy is called a versal deformation. A versal deformation is miniversal if it has the minimal number of parameters. This number is the codimension of the degenerate system.

The study of versal deformations of degenerate systems is therefore a major problem of research in bifurcation theory. (See Chapter 3 for precise definitions.)

Many results have been proved in local bifurcation theory which is concerned with degenerate singularities of vector fields or diffeomorphisms [4] [12]. In the global theory major theorems have been obtained for two dimensional dynamical systems [4] [22], but little progress has been made in higher dimensions due to existence of strange attractors and other chaotic phenomena. More fruitful in terms of application to the real world has been the study of bifurcation of dynamical systems under constraint [9]. Following Thom's work [23] and his emphasis on the importance of bifurcation theory in the mathematical modeling of physical problems, catastrophe theory, relating to dynamical systems governed by a potential function, has rapidly developed with its enormous power of application mainly in the hands of Zeeman [24]. Hamiltonian systems and dynamical systems invariant under a group of symmetry have also been an active area of research with many interesting developments [5] [11] [12].

The replicator system is a system under constraint which was introduced in population dynamics by Akin, Hofbauer, Jonker and Zeeman. The first order replicator system which is the subject of this thesis is of the form:

$$\dot{x}_i = x_i((Ax)_i - xAx) \quad , \quad x \in \mathbb{R}^{n+1} \; , \; A \in M_{n+1} \; , \quad i = 1,\dots,n+1$$

where M_{n+1} is the set of all $(n+1) \times (n+1)$ real matrices and x ambiguously denotes the column or row vector with entries x_i. This system arises in many branches of population dynamics where different species or strategies are competing with each other. If there are $n+1$ strategies, say, of type $i = 1,\dots,n+1$, with density x_i such that the pay-off to i for playing against j can be assumed to be given by the real number a_{ij}, then the time evolution of the population, under reasonable assumptions, will be given by the above system of differential equations restricted to the invariant simplex $\sum_{i=1}^{n+1} x_i = 1$, $x_i \geq 0 \;\; \forall i$, with $A = (a_{ij})$. [25]

Many basic properties of the replicator system were proved by Hofbauer, Zeeman and others [21]. In [25], Zeeman proposed a programm for classifying the replicator system as follows. Two matrices in M_{n+1} are said to be equivalent if they induce equivalent flows on the simplex, where equivalence of two flows has the usual meaning except that the homeomorphism inducing the equivalence is also required to send any k-face of the simplex to a k-face $(0 \leq k \leq n)$. One then seeks to find all the stable classes of the system in different dimensions. For $n = 1$, Zeeman proved that there are up to time reversal two stable classes; for $n = 2$, he conjectured that there are up to time reversal 19 stable classes. The basic ingredient for this conjecture was the assumption that there were no stable limit cycles for $n = 2$. This assumption was later justified when Hofbauer proved the equivalence of the replicator system with the Lotka-Volterra system. The latter does not admit stable limit cycles in two dimensions. Zeeman's conjecture was then completely proved by Carvalho who in her thesis actually showed the equivalence of any two matrices in the same conjectured class. For $n \geq 3$, stable limit cycles do occur [25] and, what is more, chaotic behaviour has been detected by Arneodo et al [3] so that no real hope of a topological classification can remain.

In this thesis three main original results are obtained:

(i) The codimension one bifurcations of the planar replicator system have been determined and classified by proving stability (miniversality) [proposition 2.3. and theorems 4.5, 6.3, 6.7, and 6.10].

(ii) The codimension two bifurcations (of the planar replicator system) have been determined, without however proving the stability (miniversality) [proposition 7.1].

(iii) The conjugacy classes of certain families of maps of intervals have been determined [Theorem 5.2], and the equivalence of certain families of vector fields has been established. [Theorem 5.6.]

An alternative and simpler proof for Carvalho's main result in her thesis is also established as a Corollary to result (i) above.

A few remarks on these results are in order. First we note that stable saddle connections, which do not appear in general dynamical systems, are a persistent feature of the replicator system and hence the standard technique of using time to construct topological equivalence fails in this system because of the existence of moduli [17]. We have therefore used arc_length instead of time for the construction of topological equivalence in this work. Although the method used has been developed by me independently, it is in fact a refined version of the techniques in Peixoto's classic papers [19] [20] developed so as to take into account the existence of saddle connections and applied so that the construction depends continuously on the parameter. (See Chapter 4.)

However the major difficulties involved in proving result (i) were two_fold. Firstly, there was the problem of the degenerate Hopf bifurcations which occur in the planar replicator system (see Chapter 3). These degenerate Hopf bifurcations have ∞ codimension in the space of all one parameter families of vector fields but they do naturally occur in many constrained systems including in perturbation of Hamiltonian systems. In contrast to the generic Hopf bifurcation, the equivalence of such degenerate Hopf bifurcations has not been proved in the literature and therefore our results in Chapter 5 (proposition 5.5 and theorem 5.6), stated in result (iii) above, are original. These results are used in the proof of result (i). Secondly, there was the problem of existence of cycle of saddles in the region where degenerate Hopf bifurcation occurs. Here the construction of topological equivalence between two families is heavily based on the use of Lyapunov functions which can be avoided only in the simplest case and that at the cost of using some fairly recent results on the linearization of a family of vector fields in the neighbourhood of a singularity (Chapter 6). In this context, we have used Lyapunov functions in a way different from their usual applications in dynamical systems.

Unless otherwise stated all the results in this thesis are mine. Some of the results are based on joint work with Professor Zeeman and I have explicity stated his contributions wherever this has been the case. The plan of the Thesis is as follows.

Chapter 1 is a summary of some of the previous results about the replicator system.

Chapter 2 is based on my M.Sc. Thesis. The parameter space of the system is reduced to a three torus and then the codimension one strata are determined.

Chapter 3 establishes the terminology in which all our results are expressed and examines the local bifurcations (degenerate Hopf and exchange of stability bifurcations) and the global bifurcations involved in the codimension one case.

Chapter 4 is concerned with codimension one bifurcations where no cycle of saddles exists. The stability (miniversality) of these bifurcations is proved.

Chapter 5 is independent of the rest of the thesis but its results are needed in the next chapter. Conjugacy classes of certain families of maps of interval and the equivalence of certain families of vector fields are established.

Chapter 6 looks at the codimension one bifurcations with cycle of saddles. The stability (miniversality) of these bifurcations is proved.

Chapter 7 determines the codimension two bifurcations and discusses the shortcomings and mistakes of Bomze in his attempt to find all the phase portraits of the planar replicator system.

Terminology and notations

The usual terminology and notations in dynamical system as for example in [18] has been used in the thesis; the terminology of bifurcation theory used here is in line with that of Arnold in [4] and [5]. In particular two vector fields are (topologically) equivalent if there exists a homeomorphism of the phase space of one onto the other which takes oriented orbits onto oriented orbits. Two parameter-dependent families of vector fields are (topologically) equivalent if there exists a homeomorphism between the parameter spaces and a family of homeomorphisms of the phase spaces depending continuously on the parameter and mapping the family of oriented orbits of the first family for every value of the parameter into the family of oriented orbits of the second family for the corresponding value of the parameter.

$\overline{PQ}$ denotes the straight line from P to Q inclusive. $\overset{\circ}{\overline{PQ}}$ denotes the interior of $\overline{PQ}$ (with P and Q removed). $\overset{\curvearrowright}{PQ}$ denotes the orbit segment from P to Q (with respect to a given flow) which may or may not include P and Q.

Chapter 1.

Definitions and basic results

In this chapter we introduce the replicator equations and state their basic properties. We will not give proofs as these can be found in [8], [15] and [25].

1.1 The Replicator equations

Let $M_{n+1} = \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ denote the space of all real $(n+1)\times(n+1)$ matrices with the usual topology. For each $A = (a_{ij}) \in M_{n+1}$ define a vector field V^A in $\mathbb{R}^{n+1}$ by

$$V_i^A(x) = x_i((Ax)_i - xAx) \quad , \quad i = 1,.., n+1$$

where x denotes ambiguously the point $x = (x_1,\dots,x_{n+1})$ of $\mathbb{R}^{n+1}$ or the column or row matrix with elements $x_1,\dots,x_{n+1}$. Then the system of differential equations

$$(*) \qquad \dot{x}_i = V_i^A(x) \qquad i = 1,..,n+1$$

induces a family of flows Δ_A on the n-dimensional invariant simplex

$$\Delta_n = \{x = (x_1,\dots,x_{n+1}) \in \mathbb{R}^{n+1} \mid x_i \geq 0 \ , \ \sum_1^{n+1} x_i = 1\} \ .$$

This simplex is the region of physical interest in population dynamics, and the replicator equations are defined as the system (*) restricted to this simplex. Note that the flow Δ_{-A} is the time reversal of Δ_A .

The replicator system is equivalent to the well known Lotka-Volterra system. More precisely Hofbauer has shown the following result.

Proposition 1.1 [25]

Under the change of coordinates $y_i = \dfrac{x_i}{x_1}$, $i = 2,..,n+1$, with $x_1 \neq 0$, the vector field (*) is equivalent to the vector field

$$\dot{y}_i = y_i\ (k_i + (By)_i)$$

where $y = (y_2,\dots,y_{n+1}) \in \mathbb{R}^n$, $k_i = a_{i1} - a_{ii}$, and $B = (b_{ij}) = (a_{ij} - a_{1j})$, $i,j = 2,\dots,n+1$. □

In line with the theory of structural stability in dynamical systems, Zeeman proposed the following natural notion of equivalence for the replicator equations [25].

Definition

$A,B \in M_{n+1}$ are said to be equivalent ($A \sim B$) if there exists a homeomorphism of Δ_n onto itself, which takes each k-dimensional face of Δ_n onto a k-dimensional face ($n \geq k \geq 0$) and maps Δ_A-orbits onto Δ_B-orbits preserving the orientation of orbits. □

Clearly the relation $\sim$ is an equivalence relation in M_{n+1} .

Identifying the matrix A with the vector field V^A we see that the above notion of equivalence which requires faces to be preserved is stronger than the usual notion of equivalence in dynamical systems.

We say that $A \in M_{n+1}$ is stable if it has a neighbourhood of equivalents in M_{n+1} and we call a property of A robust if it is shared by all matrices in a neighbourhood of A in M_{n+1}. Later on we will only be concerned with $n = 2$. Therefore, in addition to the basic properties of the replicator system which remain true for any n, we also state in the next section results which are valid only for $n = 2$.

1.2 Basic properties of the system

The simplex Δ_n has $n+1$ vertices X_i, $i = 1,\ldots,n+1$, corresponding to the points $x_i = 1$, $x_j = 0$, $j \neq i$. It has also edges $\overline{X_iX_j}$, $i < j$, corresponding to the segments $x_k = 0$, $k \neq i,j$. We write

$\overset{o}{\Delta}_n = \{x \in \Delta_n \mid x_i \neq 0 \ , \ i = 1,\ldots,n+1\}$ and

$\overset{o}{\overline{X_iX_j}} = \{x \in \overline{X_iX_j} \mid x_i \neq 0 \text{ and } x_j \neq 0\}$. When $n = 2$, we write Δ for Δ_2, and $\partial\Delta$ for $\overline{X_1X_2} \cup \overline{X_2X_3} \cup \overline{X_1X_3}$.

The parameter space M_{n+1} is $(n+1)^2$ dimensional but this can readily be reduced to $n(n+1)$ by proposition 1.2 below for which the following definitions are needed.

Definition

(1) $Z_{n+1} = \{A \in M_{n+1} \mid a_{ii} = 0 \ , \ i = 1,\ldots.,n+1\}$

(2) $Z^{+}_{n+1} = \{A \in Z_{n+1} \mid a_{ij} \neq 0 \, , \, i \neq j\}$

(3) $K_{n+1} = \{A \in M_{n+1} \mid a_{ij} = a_{ik}, i,j,k = 1,\ldots,n+1\}$. □

Then $M_{n+1} = Z_{n+1} \oplus K_{n+1}$ and we have:

<u>Proposition 1.2</u> [25]

(i) For $A,B \in M_{n+1}$, $\Delta_A = \Delta_B$ iff $A-B \in K_{n+1}$.

(ii) {Equivalent classes in M_{n+1}} =

{Equivalent classes in Z_{n+1}} $\oplus K_{n+1}$. □

We can therefore work with Z_{n+1} which has $n(n+1)$ dimensions. In proposition 1.3, the main results about the fixed points of the replicator system are collected.

<u>Proposition 1.3</u> [25]

Let $A = (a_{ij}) \in Z_{n+1}$ and $\beta = \beta(A) = \text{adj}(A)u$ where (adj A) is the adjoint of A and u the column vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$, then

(i) All the faces of Δ_n are invariant. In particular the vertices X_i are fixed.

(ii) The eigenvalue of Δ_A at X_i corresponding to the eigenvector in the direction $\overline{X_iX_j}$ is a_{ji} .

(iii) If $A \in Z_{n+1}$ is stable then $A \in Z^+_{n+1}$.

(iv) If there are two fixed points in $\overset{\circ}{\overline{X_iX_j}}$ then $\overline{X_iX_j}$ is point wise fixed.

(v) There exists an isolated fixed point in $\overset{\circ}{\overline{X_iX_j}}$ iff $a_{ij}a_{ji} > 0$.

(vi) If q is an isolated fixed point in $\overset{\circ}{\overline{X_iX_j}}$ then q is unique and this property is robust.

(vii) [n = 2] The eigenvalues of an isolated fixed point q in $\overset{\circ}{\overline{X_iX_j}}$ $\left(q_i = \frac{a_{ij}}{a_{ij}+a_{ji}}, q_j = \frac{a_{ji}}{a_{ij}+a_{ji}}\right)$ are $\frac{-a_{ij}a_{ji}}{a_{ij}+a_{ji}}$ and $\frac{\beta_k}{a_{ij}+a_{ji}}$ $(k \neq i,j)$ for the eigenvectors in the direction of $\overline{X_iX_j}$ and in the transversal direction respectively.

(viii) If there are two fixed points in $\overset{\circ}{\Delta}_n$ then the line joining them is pointwise fixed.

(ix) If p is an isolated point in $\overset{\circ}{\Delta}_n$ then p is unique and this property is robust. Moreover in this case $\beta = (\mathrm{adj}\ A)u$ will have all components positive or all negative and $p = \overset{\circ}{\Delta}_n \cap [(\mathrm{adj}\ A)u]$ where $[(\mathrm{adj}\ A)u]$ denotes the subspace of $\mathbb{R}^{n+1}$ generated by $(\mathrm{adj}\ A)u$.

(x) If A is stable then Δ_A has at most one fixed point in $\overset{\circ}{\Delta}_n$. □

From now on throughout this thesis assume $n = 2$. When A is stable and has a fixed point in $\overset{o}{\Delta}$, the eigenvalues at the fixed point are most easily calculated if this fixed point is the barycentre of Δ. This motivates the following definition. Call $A \in Z_3$ central if Δ_A has an isolated fixed point at the barycentre of Δ. By proposition 2(ix), A is central iff the sum of entries of its rows are equal and we have:

Proposition 1.4 [25]

If A is central, $A = \begin{pmatrix} 0 & \theta+a_1 & \theta-a_1 \\ \theta-a_2 & 0 & \theta+a_2 \\ \theta+a_3 & \theta-a_3 & 0 \end{pmatrix}$

say, then the eigenvalues of Δ_A at the barycentre are given by the roots of the quadratic equation

$$\lambda^2 + \frac{2\theta\lambda}{3} + \frac{\theta^2+\rho}{9} = 0 \quad \text{where} \quad \rho = \sum_{i<j} a_i a_j \ . \qquad \square$$

Now let $A \in Z_3$ be any stable matrix such that Δ_A has a fixed point in $\overset{o}{\Delta}$, then the following lemma and its corollary shows that A is equivalent to a central matrix from which the eigenvalues of the fixed point can be calculated.

Lemma 1.5 [25]

Let $P = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix}$, $p_i > 0$ $(i = 1,2,3)$, be a positive diagonal matrix and let $\hat{p} : \Delta \to \Delta$ be the diffeomorphism given by

$(\hat{p}x)_i = \dfrac{p_i x_i}{\sum_j p_j x_j}$. Then $\hat{p}$ induces an equivalence between A and AP . □

Corollary 1.6

If $\bar{x} \in \mathring{\Delta}$ is a fixed point of Δ_{AP} with eigenvalues λ_1 and λ_2 then $\hat{p}(\bar{x})$ is a fixed point of Δ_A with eigenvalues

$\dfrac{\lambda_1}{\Sigma p_i \bar{x}_i}$ and $\dfrac{\lambda_2}{\Sigma p_i \bar{x}_i}$. □

It follows that if $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \mathring{\Delta}$ is a fixed point of Δ_A then the barycentre $E = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a fixed point of Δ_{AP} with $p_i = \dfrac{1}{3\bar{x}_i}$. We call AP the centralization of A .

Having obtained explicit expressions for the eigenvalues of the fixed points of Δ_A , one can show that if Δ_A has a nonhyperbolic fixed point then perturbations of A can result in different types of fixed points near the original one, i.e.

Proposition 1.7 [8]

If $A \in Z_3$ is stable then all the eigenvalues of fixed points of Δ_A are hyperbolic. □

We also need to study the limit cycles of the system. Unstable closed orbits can occur in the planar equations as we will study in later chapters. However an application of De Luc's lemma [1] to the planar Lotka-Volterra equations proves that stable limit cycles do not occur in them. Hence we have:

Proposition 1.8 [8]

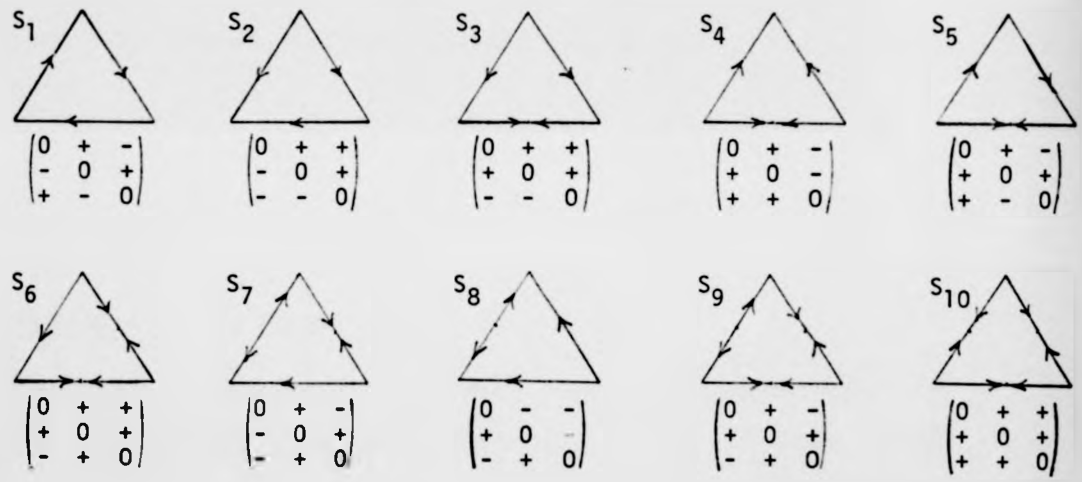
Stable limit cycles do not occur in the planar replicator system. □

1.3 Decomposition of the parameter space

The first step in decomposing the parameter space into stable classes exploits the invariance of $\partial\Delta$ under equivalence, which implies that the phase portraits of the flows on $\partial\Delta$ induced by two equivalent matrices are the same up to a permutation of vertices. However, by proposition 1.3 (i)-(iii), the phase portrait on $\partial\Delta$ for a stable matrix is determined by the signs of its entries which must therefore be invariant under equivalence up to a permutation of indices. This leads to the following definition. Let Ω_3 denote the permutation group of $\{1,2,3\}$ and write $\bar{\sigma}A$ for the matrix obtained by permuting both rows and columns of A by $\sigma \in \Omega_3$. Then the map

$$\begin{cases} \tilde{\sigma} : \Delta \to \Delta \\ \tilde{\sigma} : x \to \tilde{\sigma}(x) \end{cases}$$

with $(\tilde{\sigma}(x))_i = x_{i\sigma}$ induces an equivalence between A and $\bar{\sigma}A$. Say $A,B \in Z_3^+$ are sign equivalent if their off-diagonal elements have the same sign i.e. $a_{ij}\, b_{ij} > 0$, $\forall\, i,j\ (i \neq j)$. Say A,B $Z_3^{\pm}$ are <u>combinatorially equivalent</u> if there exists $\sigma \in \Omega_3$ such that $\bar{\sigma}A$ and B are sign equivalent. One then shows that stable classes refine combinatorial classes and that there are up to time reversal 10 combinatorial classes [25]. In Figure 1.1 an example of a sign class S_m is given for each combinatorial class c_m $(m = 1,\dots,10)$.



<u>Figure 1.1</u>

For two matrices to be equivalent, their fixed points in the interior of edges and in $\mathring{\Delta}$ must also be pairwise of the same type. This further decomposes each combinatorial class into regions where these fixed points are of the same type. There are, up to time reversal, 19 regions as such which were obtained by Zeeman, who was assuming that limit cycles do not occur in the system, a conjecture which was later proved by

Hofbauer (proposition 1.8). It remained to prove that each of these 19 regions in fact correspond to a class i.e. that any two matrices in the same region are equivalent. A complete proof of this was given by Carvalho who, for proving this, developed a method for constructing topological equivalence between quasi-gradient flows with the same circular distribution: (We will present a simpler proof of this equivalence in the thesis.) The decomposition of the parameter space into stable classes was then completed. It is given in Theorem 1.9 below.

Let S_i , $i = 1,\dots,10$, be the sign classes as in Figure 1.1, and for $K \subset Z_3$ let $G(K) = \{A \in Z_3 \mid A$ is combinatorially equivalent to an element of $K\}$. Then

Theorem 1.9 [8] and [25]

(i) $A \in S_1$ is stable iff $\det A \neq 0$. Define the stable class (1) in Z_3 by $(1) = G\{A \in S_1 \mid \det A > 0\}$. If (-1) denotes the time reversal class of (1) we then have $(-1) = G\{A \in S_1 \mid \det A < 0\}$.

(ii) $A \in S_2$ is stable and we define $(2) = C_2$.

(iii) $A \in S_3$ is stable and we define $(3) = C_3$.

(iv) $A \in S_4$ is stable iff $\beta_3 \neq 0$. (Recall $\beta = (\operatorname{adj} A)u$ where $u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$) Define the two classes $(4_1) = G\{A \in S_4 \mid \beta_3 < 0\}$ and $(4_2) = G\{A \in S_4 \mid \beta_3 > 0\}$.

(v) $A \in S_5$ is stable iff $\beta_3 \neq 0$. Define $(5_1) = G\{A \in S_5 | \beta_3 > 0\}$ and $(5_2) = G\{A \in S_5 | \beta_3 < 0\}$.

(vi) $A \in S_6$ is stable iff $\beta_2 \neq 0$ and $\beta_3 \neq 0$. Define $(6_1) = G\{A \in S_6 | \beta_1 > 0, \beta_3 > 0\}$, $(6_2) = G\{A \in S_6 | \beta_1 < 0, \beta_3 > 0\}$, $(6_3) = G\{A \in S_6 | \beta_1 > 0, \beta_3 < 0\}$ and $(6_4) = \{A \in S_6 | \beta_1 < 0, \beta_3 < 0\}$.

(vii) $A \in S_7$ is stable iff $\beta_2 \neq 0$ and $\beta_1 \neq 0$ and $\det A \neq 0$ when $\beta_1 > 0$ and $\beta_2 > 0$. Define $(7_1) = G\{A \in S_7 | \beta_1 > 0 , \beta_2 > 0 , \det A > 0\}$, $(7_2) = G\{A \in S_7 | \beta_1 < 0 , \beta_2 < 0\}$ and $(7_3) = G\{A \in S_7 | \beta_1 < 0 , \beta_2 > 0\}$. Then we have $(-7_1) = G\{A \in S_7 | \beta_1 > 0, \beta_2 > 0 , \det A < 0\}$ and $(-7_3) = G\{A \in S_7 | \beta_1 > 0, \beta_2 < 0\}$.

(viii) $A \in S_8$ is stable and we define $(8) = C_8$.

(ix) $A \in S_9$ is stable iff $\beta_1 \neq 0$ and $\beta_2 \neq 0$. Define $(9_1) = G\{A \in S_9 | \beta_1 < 0, \beta_2 < 0\}$ and $(9_2) = G\{A \in S_9 | \beta_1\beta_2 < 0\}$.

(x) $A \in S_{10}$ is stable iff $\beta_i \neq 0$, $i = 1,2,3$. Define $(10_1) = G\{A \in S_{10} | \beta_i > 0,\ i = 1,2,3\}$ and $(10_2) = G\{A \in S_{10} | \beta_i < 0 \text{ some } i\}$. □

In Figure 1.2 an example of each stable class, up to time reversal, is sketched. Attractors are marked with a solid dot, repellors by an open dot, and saddles by their insets and outsets. All other orbits flow from a repellor to an attractor, except in class (1) where the α-limit set of any point in $\mathring{\Delta}\backslash E$ is $\partial\Delta$.

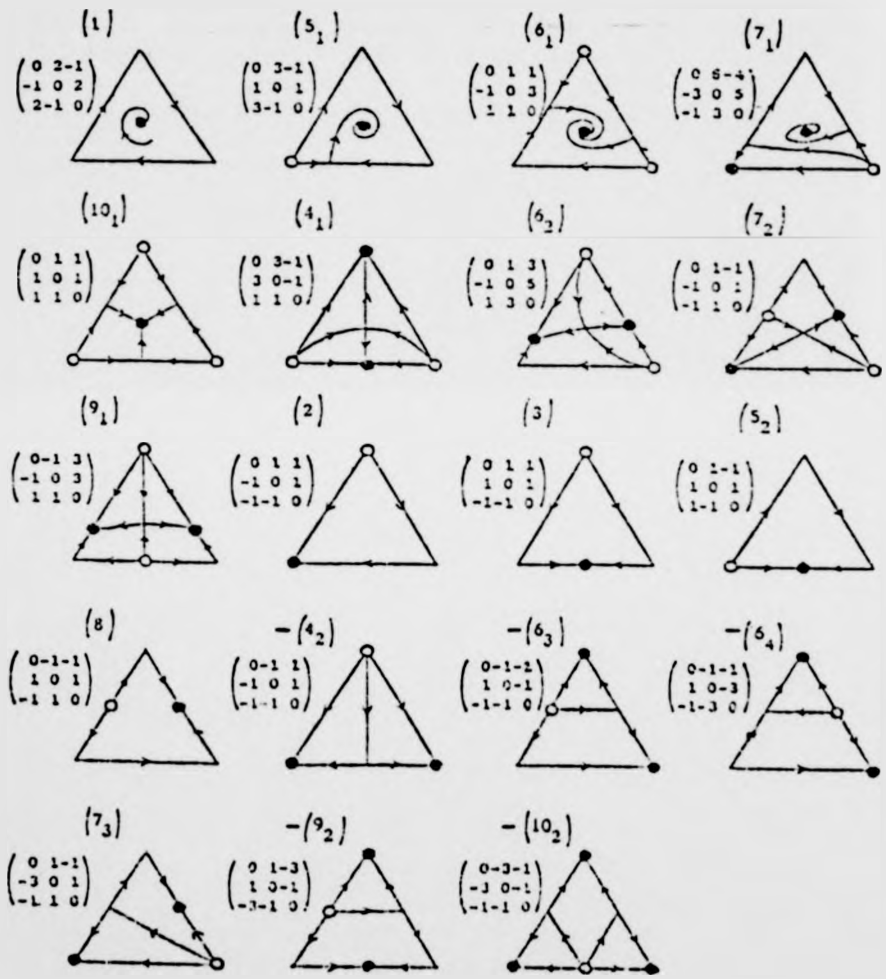


Figure 1.2

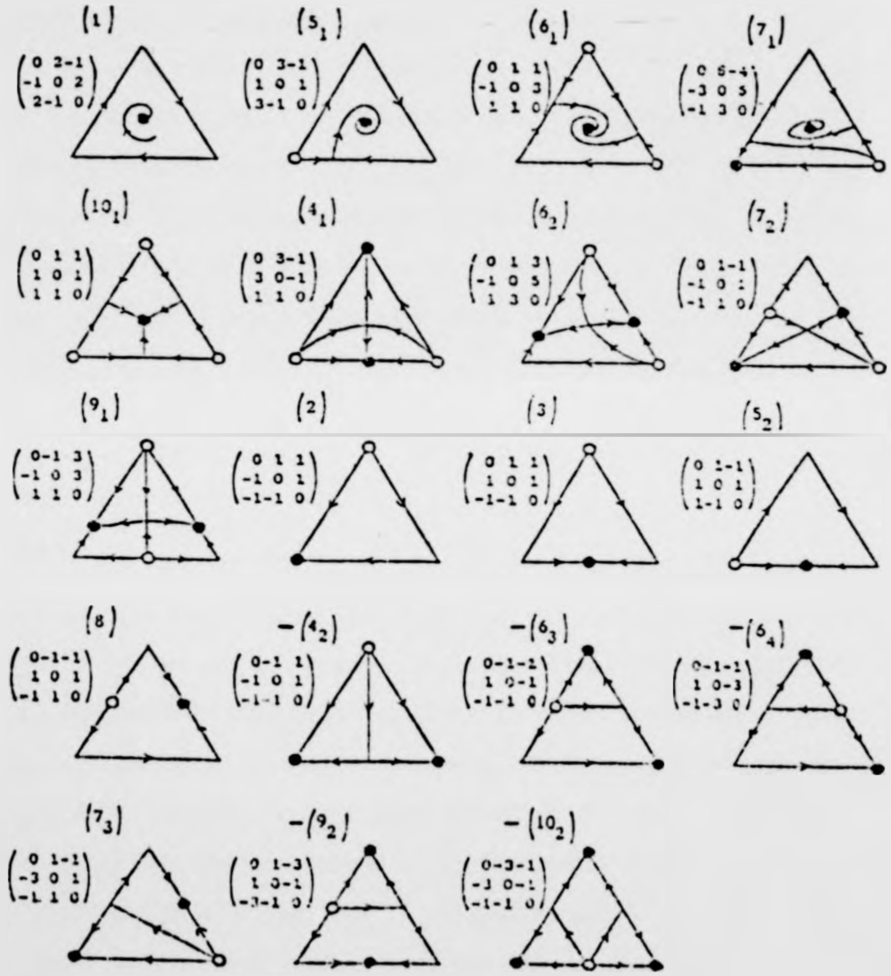


Figure 1.2

Chapter 2.

Reduction of the parameter space

In this chapter we will look more closely at the parameter space Z_3 and by reducing it to a three dimensional parameter space we will study how the stable classes are joined in the parameter space, which will enable us to determine the codimension one strata in this space. The material in this chapter is based on my M.Sc. Thesis [10], where the proofs are presented in more details.

2.1 Stratification of Z_3

The decomposition of $Z_3 \cong \mathbb{R}^6$ in Theorem 1.9 into stable classes by the hypersurfaces $(A)_{ij} = a_{ij} = 0 \; (i \neq j)$,
$(\beta(A))_i = -a_{i+1\, i+2}\, a_{i+2\, i+1} + a_{i\, i+1}\, a_{i+1\, i+2} + a_{i\, i+2}\, a_{i+2\, i+1} = 0$
(indices are mod 3) and $\det A = a_{12}a_{23}a_{32} + a_{13}a_{21}a_{32} = 0$ is a stratification of Z_3 i.e. a partition into finitely many smooth disjoint submanfiolds (strata) given by algebraic equations and inequalities [4]. For convenience, we write the codimension of a matrix with respect to this stratification as cod. (In chapter 4, we will introduce another notion of codimension.) In this work we are only concerned with the cod 1 stratum except in the last chapter where we shall look at the cod 2 stratum. If A is a cod 1 matrix, then it lies on exactly one of the hypersurfaces above. It is also easy to check that such A is always a non singular point of the hypersurface

to which it belongs. However, to proceed further with the six-dimensional parameter space and to try to partition the cod 1 stratum to its different components (strata), according to the pair of stable classes nearby, is a formidable task. Fortunately this is not needed as a further reduction of the parameter space is possible. This reduction in the next section is due to Zeeman.

2.2 The Three Torus Q

Let Q be the three torus $\mathbb{R}^3/(2\pi\mathbb{Z})^3$ and define a map

$$\begin{cases} W : Q \to Z_3 \\ \quad \alpha \to W(\alpha) \quad , \qquad \alpha = (\alpha_1,\alpha_2,\alpha_3) \in Q \end{cases}$$

where $$W(\alpha) = \begin{pmatrix} 0 & \operatorname{Sin}(\alpha_2 - \frac{\pi}{3}) & \operatorname{Sin}\alpha_3 \\ \operatorname{Sin}\alpha_1 & 0 & \operatorname{Sin}(\alpha_3 - \frac{\pi}{3}) \\ \operatorname{Sin}(\alpha_1 - \frac{\pi}{3}) & \operatorname{Sin}\alpha_2 & 0 \end{pmatrix}$$

W is clearly one to one and continuous. The significance of $\frac{\pi}{3}$ in the definition of W will become clear later. Let

$$\mathbb{R}^{3^+} = \{r = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \in \mathbb{R}^3 \mid r_i > 0 \ , \ i = 1,2,3\} \ .$$ Then

$\operatorname{Im} W \otimes \mathbb{R}^{3^+} = \{W(\alpha) \times r \mid \alpha \in Q \ , \ r \in \mathbb{R}^{3^+}\}$ will be the set of all matrices in Z_3 that in each column have at most one zero off-diagonal entry. We can now deduce:

Proposition 2.1

$$\{\text{stable classes in } Z_3\} = \{\text{stable classes in } \operatorname{Im} W\} \times \mathbb{R}^{3^+} .$$

Proof

By proposition 1.3(iii), {stable classes in Z_3} = {stable classes in Z_3^+} . But $Z_3^+ \subset \operatorname{Im} W \times \mathbb{R}^{3^+} \subset Z_3$ so {stable classes in Z_3^+} = {stable classes in $\operatorname{Im} W \times \mathbb{R}^{3^+}$} . Now lemma 1.5 implies: {stable classes in $\operatorname{Im} W \times \mathbb{R}^{3^+}$} = {stable classes in $\operatorname{Im} W$} $\times \mathbb{R}^{3^+}$. □

The problem of stratification of the parameter space into stable classes therefore reduces to stratifying the set $\operatorname{Im} W$ or equivalently the 3-torus Q . To do this we identify $\alpha \in Q$ with $W(\alpha) \in Z_3$ and thereby extend the notions of sign class, combinatorial class and stable class to Q , e.g. $\alpha \in 6_2$ means $W(\alpha) \in 6_2$.

We start by studying the combinatorial symmetry in Q . Define an action of Ω_3 , the permutation group of $\{1,2,3\}$, on Q by $(ij)\alpha = \alpha'$, $i \neq j$, where

$$\alpha'_k = \frac{4\pi}{3} - \alpha_k \qquad k \neq i,j$$

$$\alpha'_i = \frac{4\pi}{3} - \alpha_j \quad \text{and} \quad \alpha'_j = \frac{4\pi}{3} - \alpha_i \quad .$$

Then it is easy to check that $(1)\alpha = \alpha$ and for distinct i,j,k we have $(ijk)\alpha = \alpha'$ where $\alpha'_\ell = \alpha_{\ell(ijk)}$. Furthermore one can show by direct manipulation that the following diagram commutes (see my M.Sc. Thesis [10]):

$$\begin{tikzcd} Q \arrow[r, "W"] \arrow[d, "\sigma"'] & Z_3 \arrow[d, "\bar{\sigma}"] \\ Q \arrow[r, "W"] & Z_3 \end{tikzcd}$$

i.e. The action of Ω_3 on Q gives the combinatorial symmetry of Q. The time reversal symmetry in Q is obtained by noting that $W(\alpha+\pi) = -W(\alpha)$ where $(\alpha+\pi)_i = \alpha_i+\pi$.

Next note that the twelve planes $\alpha_i = 0, \frac{\pi}{3}, \pi, \frac{4\pi}{3}$ $(i = 1,2,3)$ partition Q into 4^3 cubic regions each of which lies in a sign class. A straightforward calculation in [10] also shows that we obtain the expression $(\beta(W(\alpha)))_i = \frac{\sqrt{3}}{2}\operatorname{Sin}(\alpha_{i+1} - \alpha_{i+2} + \frac{\pi}{3})$, the simple form of which is due to the choice of $\frac{\pi}{3}$ in the definition of W.

Hence the hypersurfaces $\beta_i = 0$, $1,2,3$, reduce to the planes $\alpha_{i+1} - \alpha_{i+2} + \frac{\pi}{3} = 0 \pmod{\pi}$, $i = 1,2,3$, and we can check from Theorem 1.9 that in fact only the three planes $\alpha_{i+1} - \alpha_{i+2} + \frac{\pi}{3} = 0$ $(i = 1,2,3)$ contribute to the stratification. We also observe that the hypersurface $\det A = 0$ reduces to the surface $\det W(\alpha) = \operatorname{Sin}\alpha_1 \operatorname{Sin}\alpha_2 \operatorname{Sin}\alpha_3 + \operatorname{Sin}(\alpha_1 - \frac{\pi}{3})\operatorname{Sin}(\alpha_2 - \frac{\pi}{3})\operatorname{Sin}(\alpha_3 - \frac{\pi}{3}) = 0$ which in each cubic component of C_1 in Q joins three pairs of the opposite edges (Figure 2.1), and in each cubic component of C_7 separates (7_1) and (-7_1) from each other by joining four edges of the tetrahedron containing (7_1) and (7_2) in a saddle shape (Figure 2.2).

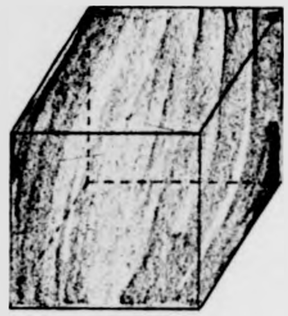


Figure 2.1

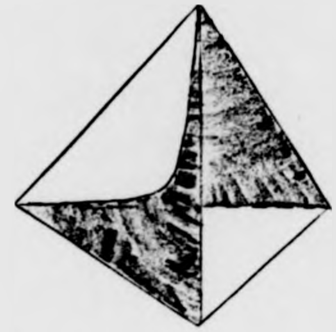


Figure 2.2

Collecting all this information and using Theorem 1.9 we have deduced:

Proposition 2.2

The stratification of Q into stable classes is determined by the planes $\alpha_i = 0 , \frac{\pi}{3} , \pi , \frac{4\pi}{3}$ $(i = 1,2,3)$, the planes $\alpha_i - \alpha_{i+1} + \frac{\pi}{3} = 0$ and the surfaces $\det W(\alpha) = 0$ in C_1 and C_7 . □

The stratification of Q is sketched in Figures 2.3 and 2.4. These are obtained by using the algebraic conditions for the stable classes extracted from Theorem 1.9 and applied to Q by proposition 2.2. The combinatorial and time reversal symmetries of Q are then used to complete the stratification. Figure 2.3 depicts the layer of Q with $0 \le \alpha_1 \le \frac{\pi}{3}$ while Figure 2.4 corresponds to $\frac{\pi}{3} \le \alpha_1 \le \pi$. The other half of Q is the time reversal of this half under the relation $W(\alpha+\pi) = -W(\alpha)$. For Figure 2.3 we have drawn the cross-section $\alpha_1 = \frac{\pi}{6}$ and for Figure 2.4 the two sections $\alpha_1 = \frac{\pi}{2}$ and $\alpha_1 = \frac{5\pi}{6}$. A perpendicular arrow to the plane $\alpha_{i+1} - \alpha_{i+2} + \frac{\pi}{3} = 0$ indicates the side in which β_i is positive.

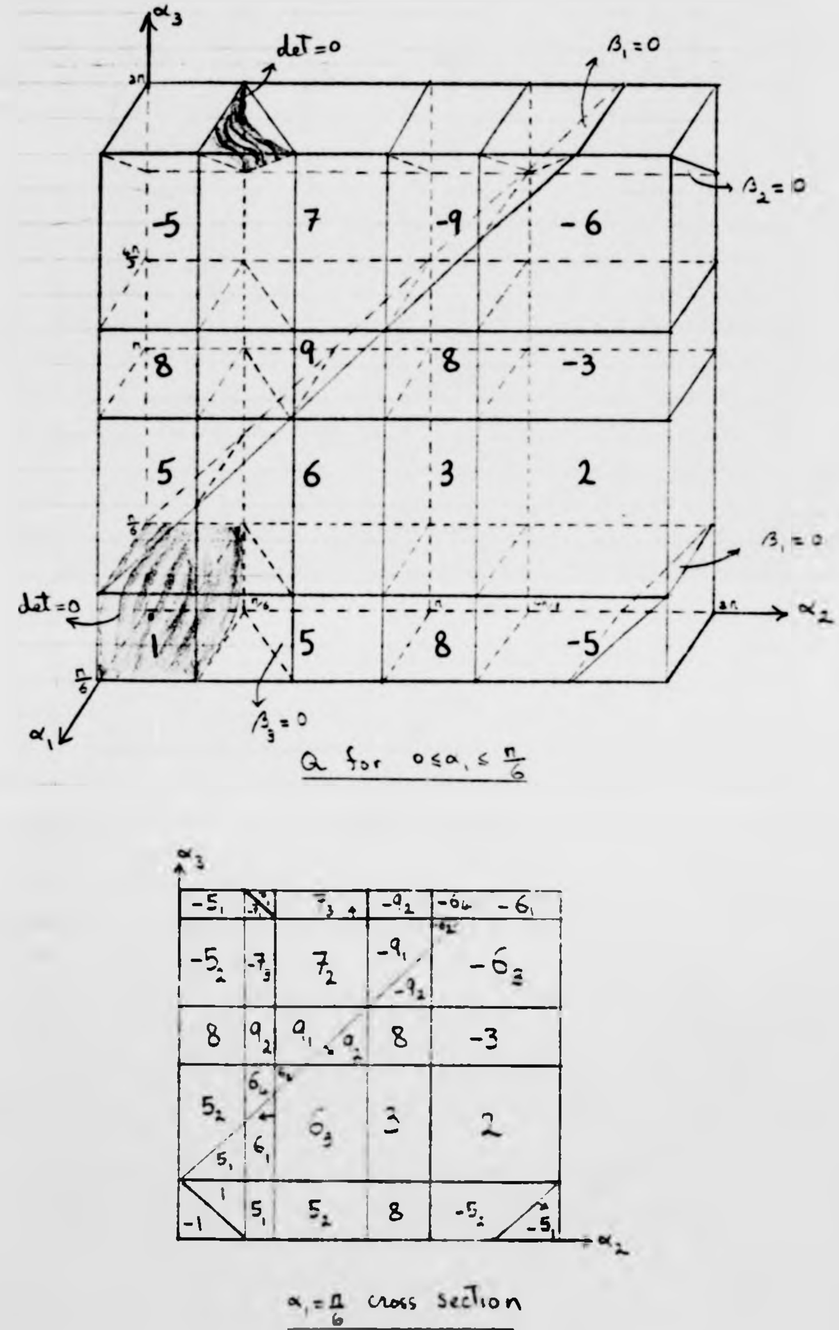
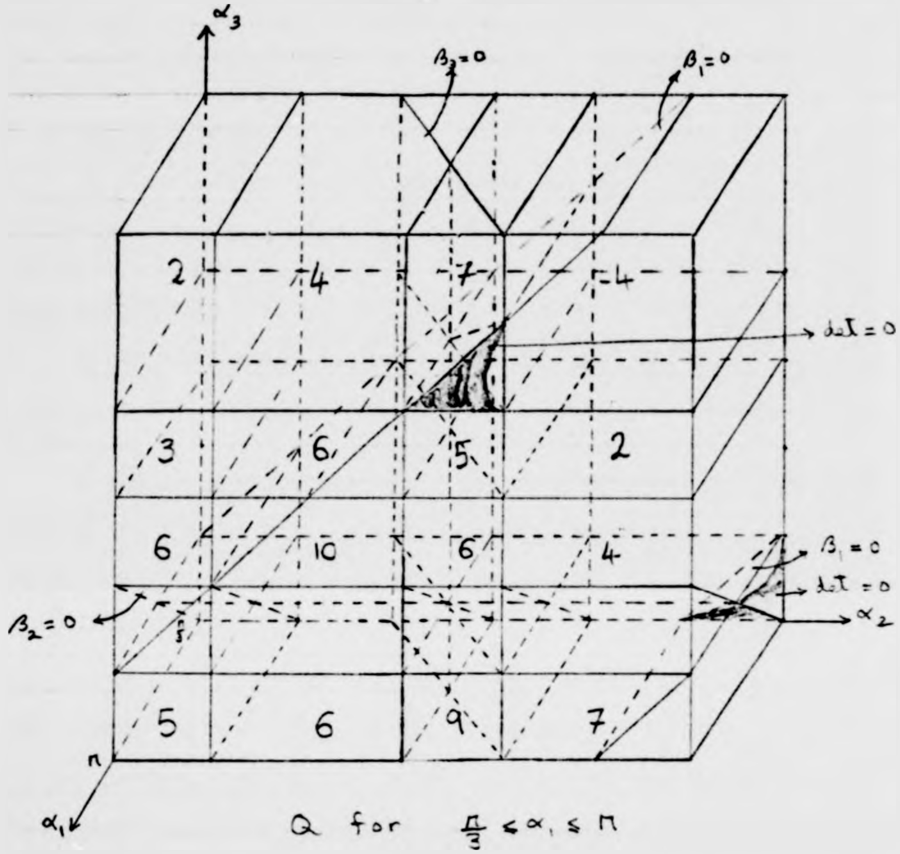
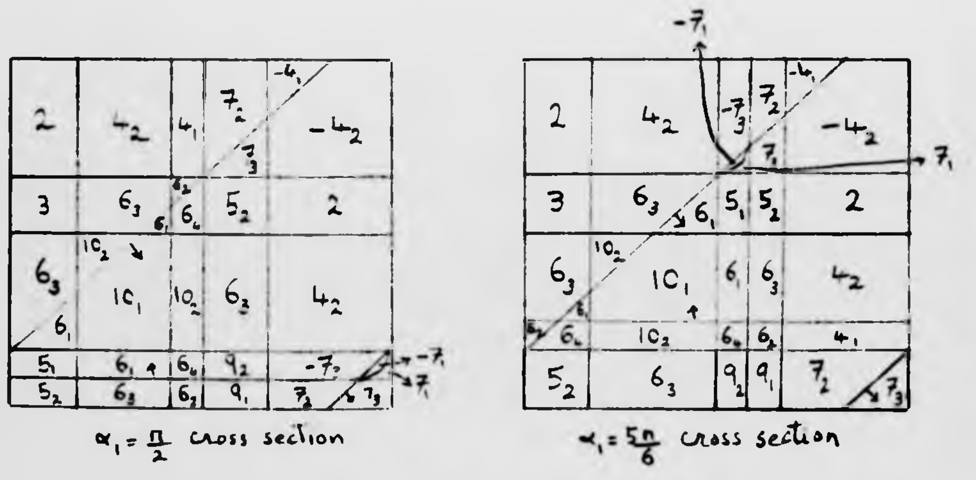

det T = 0
$\beta_1 = 0$
$\beta_2 = 0$
$\beta_3 = 0$
det = 0
α_1
α_2
α_3
Q for $0 \le \alpha_1 \le \frac{\pi}{6}$
$\alpha_1 = \frac{\pi}{6}$ cross section

Figure 2.3



α_3
$\beta_2 = 0$
$\beta_1 = 0$
$\det T = 0$
$\beta_1 = 0$
$\det T = 0$
α_2
$\beta_2 = 0$
α_1
Q for $\frac{\pi}{3} \le \alpha_1 \le \pi$



$\alpha_1 = \frac{\pi}{2}$ cross section
$\alpha_1 = \frac{5\pi}{6}$ cross section

Figure 2.4

2.3 Cod 1 strata

In this section we will determine all the cod 1 strata of different types as defined in section 2.1.

Proposition 2.3

Up to time reversal, there are 38 cod 1 strata of different types.

Proof

A cod 1 matrix has at most one zero off-diagonal entry and so it must be in $\text{Im}\, W \otimes \mathbb{R}^{3^+}$. Hence $\{$cod 1 strata in $Z_3\} = \{$cod 1 strata in $\text{Im}\, W \otimes \mathbb{R}^{3^+}\} = \{$cod 1 strata in $\text{Im}\, W\} \otimes \mathbb{R}^{3^+}$. It is then sufficient to determine the cod 1 strata in $\text{Im}\, W$ or equivalently in Q. Now proposition 2.2 implies that the cod 1 strata in Q are contained in the planes $\alpha_i = 0, \frac{\pi}{3}, \pi, \frac{4\pi}{3}$, the planes $\alpha_i - \alpha_{i+1} + \frac{\pi}{3} = 0$ and in the surfaces $\det W(\alpha) = 0$ in C_1 and in the tetrahedron in C_7. Using the three cross sections in Figures 2.3 and 2.4 we find that up to time reversal there are 38 cod 1 strata, each of which lies on the bounday of a distinct pair of stable classes. □

These 38 cod 1 strata are listed in appendix 1 according to the pair of adjacent stable classes. We have enumerated these by I_i $(i = 1,2,\ldots 38)$ so that cod 1 stratum $= \bigcup_{i=1}^{38} I_i$. To fully understand these cod 1 matrices and in particular to determine the phase portraits of the flows induced by them, we must pass from the static to the dynamic approach and examine the bifurcations involved. This we will start to do in the next chapter.

Chapter 3.

Local and Global Bifurcations

In this chapter we first set up the terminology in which all our results are expressed. This terminology is in line with that of Arnold for bifurcation theory in [4] and [5], which we have adapted to our problem. We then look at the bifurcations induced as a smooth curve intersects a cod 1 stratum transversally and we find that locally degenerate Hopf bifurcations and exchange of stability (transcritical) bifurcations are possible. The study of these will enable us to determine the phase portraits of the cod 1 flows i.e. those induced by cod 1 matrices.

3.1 Unfoldings and deformations

We start with some basic definitions:

Definition

A k-unfolding of $A \in Z_3$ is a smooth map $f : U \to Z_3$ where U is a neighbourhood of 0 in $\mathbb{R}^k$ and $f(0) = A$. The germ of such f at 0 is called a k-deformation of A. □

We write $(U,f)_k$ for a k-unfolding and $(0,f)_k$ for its germ. Given a deformation $(0,f)_k$ with a representative $f : U \to Z_3$ we call $(U,f)_k$ a representative unfolding of $(0,f)_k$. In choosing a representative unfolding we always assume U to be arbitrary small so that we are confined to an arbitrary small neighbourhood of $f(0)$ in Z_3. For convenience 1-unfoldings and 1-deformations are called unfoldings and deformations, and are denoted by (U,f) and $(0,f)$ respectively.

A deformation of $A \in Z_3$ is called transversal if it is transversal ($\pitchfork$) to all strata in Z_3. Hence a transversal deformation contains perturbations representing all the stable classes nearby. Clearly a deformation of a cod j matrix, $0 \leq j \leq 6$, is transversal iff it is transversal to the cod j stratum. Next we will define the notion of equivalent deformations.

Definition

Two deformations $(0,f)_k$ and $(0,\bar{f})_k$ are equivalent if for every pair of representative unfoldings $(U,f)_k$ and $(\bar{U},\bar{f})_k$, there exists $U^* \subset U$, $\bar{U}^* \subset \bar{U}$, a homeomorphism $\eta : U^* \to \bar{U}^*$ and a one parameter family of homeomorphisms $H_\varepsilon : \Delta \to \Delta$, $\varepsilon \in U^*$, which for each ε gives an equivalence between $f(\varepsilon)$ and $\bar{f}(\eta(\varepsilon))$ and such that H_ε depends continuously on ε. □

The deformation $(0,\bar{f})_k$ is said to be induced from $(0,f)_\ell$ if there exists germ of contonuous maps $j : \mathbb{R}^k \to \mathbb{R}^\ell$ with $j(0) = 0$ such that $\bar{f} = f \circ j$. A deformation $(0,f)_k$ is said to be a topologically versal deformation of $f(0) \in Z_3$ if every deformation of $f(0)$ is equivalent to one induced from $(0,f)_k$. Therefore a topologically versal deformation of a matrix is in fact rich enough to represent all the possible deformations of that matrix. When k is minimal $(0,f)_k$ is said to be topologically miniversal and k is called the "codimension" of $f(0)$. Hence we have two notions of co-dimension for a matrix in Z_3 : one determined by its position in the stratified Z_3 and one deduced from bifurcation theory. To avoid confusion we indicate the first notion as before by cod and the second by "codimension".

It is the object of this thesis to construct topologically versal deformations of cod 1 matrices of the planar replicator system. In fact we will prove that transversal deformations of these matrices are, in general, topologically miniversal and therefore the two notions of codimensions coincide in the codimension one case. One can also state our result in another way and say that the cod 1 bifurcations of the planar replicator system are stable in the sense that every transversal deformation of a cod 1 matrix is, in general, stable with respect to the above notion of equivalence. This latter statement explains the title of the thesis.

Let us use the terminology of this section to show that all transversal deformations of cod 1 matrices in Z_3 have their representations in $\mathrm{Im}\, W$ and Q . Let $(0,f)$ be a deformation of a cod 1 matrix $f(0) \in Z_3$. We can write f in the form

$$f(\varepsilon) = \begin{pmatrix} 0 & a_{12}(\varepsilon) & a_{13}(\varepsilon) \\ a_{21}(\varepsilon) & 0 & a_{23}(\varepsilon) \\ a_{31}(\varepsilon) & a_{32}(\varepsilon) & 0 \end{pmatrix} ,$$ or alternatively in the form

$$f(\varepsilon) = \begin{pmatrix} 0 & r_2(\varepsilon)\mathrm{Sin}(\alpha_2(\varepsilon)-\frac{\pi}{3}) & r_3(\varepsilon)\mathrm{Sin}\,\alpha_3(\varepsilon) \\ r_1(\varepsilon)\mathrm{Sin}\,\alpha_1(\varepsilon) & 0 & r_3(\varepsilon)\mathrm{Sin}(\alpha_3(\varepsilon)-\frac{\pi}{3}) \\ r_1(\varepsilon)\mathrm{Sin}(\alpha_1(\varepsilon)-\frac{\pi}{3}) & r_2(\varepsilon)\mathrm{Sin}\,\alpha_2(\varepsilon) & 0 \end{pmatrix}$$

where a_{ij} , r_i and α_i are smooth germs. First we assert that $(0,f)$ is equivalent to $(0,\bar{f})$ where

$$\bar{f}(\varepsilon) = \begin{pmatrix} 0 & \mathrm{Sin}(\alpha_2(\varepsilon)-\frac{\pi}{3}) & \mathrm{Sin}\,\alpha_3(\varepsilon) \\ \mathrm{Sin}\,\alpha_1(\varepsilon) & 0 & \mathrm{Sin}(\alpha_3(\varepsilon)-\frac{\pi}{3}) \\ \mathrm{Sin}(\alpha_1(\varepsilon)-\frac{\pi}{3}) & \mathrm{Sin}\,\alpha_2(\varepsilon) & 0 \end{pmatrix} .$$

This is because given any pair of representative unfoldings (U,f) and $(\bar{U},\bar{f})$ we can choose $U^* \subset U \cap \bar{U}$ small enough such that $r_i(\varepsilon) \neq 0$ for $\varepsilon \in U^*$, $i = 1,2,3$. Then the family of diffeomorphisms

$H_\varepsilon : \Delta \to \Delta$ with $(H_\varepsilon(x))_i = \dfrac{r_i(\varepsilon)x_i}{\sum_j r_j(\varepsilon)x_j}$, which by lemma 1.5 gives

for each $\varepsilon \in U$ an equivalence between $f(\varepsilon)$ and $\bar{f}(\varepsilon)$, will depend continuously on ε (here η = identity), and our assertion is proved.

Next we claim that $(0,f)$, $(0,\bar{f})$ and $(0,\alpha)$ are transversal if any of them is transversal (here $(0,\alpha)$ is the germ induced by $(0,f)$ with codomain Q). There are three cases corresponding to (i) $f(0) \in \{A \mid a_{ij} = 0\}$, (ii) $f(0) \in \{A \mid (\beta(A))_i = 0\}$ and (iii) $f(0) \in \{A \mid \det A = 0\}$. (For simplicity we will always refer to these hypersurfaces as $\{a_{ij} = 0\}$, $\{\beta_i = 0\}$ and $\{\det = 0\}$, respectively). Here we verify the claim for the case (ii) as the other two are entirely similar. We have $(0,f) \pitchfork \{\beta_i = 0\}$ <=> $\frac{d}{d\varepsilon}\beta_i(\varepsilon)\big|_{\varepsilon=0} \neq 0$ <=> $\frac{d}{d\varepsilon}\frac{\sqrt{3}}{2}\, r_{i+1}(\varepsilon)\, r_{i+2}(\varepsilon)\, \mathrm{Sin}(\alpha_{i+1}(\varepsilon) - \alpha_{i+2}(\varepsilon) + \frac{\pi}{3})\big|_{\varepsilon=0} \neq 0$ <=> $\frac{d}{d\varepsilon}\,\mathrm{Sin}(\alpha_{i+1}(\varepsilon) - \alpha_{i+2}(\varepsilon) + \frac{\pi}{3})\big|_{\varepsilon=0} \neq 0$ <=> $\frac{d}{d\varepsilon}\,(\alpha_{i+1}(\varepsilon) - \alpha_{i+2}(\varepsilon) + \frac{\pi}{3})\big|_{\varepsilon=0} \neq 0$. The fourth inequality in the chain implies that $(0,\bar{f}) \pitchfork \{\beta_i = 0\}$ while the last implies that $(0,\alpha) \pitchfork \{\alpha_{i+1} - \alpha_{i+2} + \frac{\pi}{3} = 0\}$, proving our claim. It therefore follows that transversal deformations of cod 1 matrices in Z_3 have their equivalents in Im W and Q . Based on the space Q , appendix 1 presents a transversal deformation for each of the cod 1 strata I_i (i = 1,....,38).

3.2 Degenerate Hopf Bifurcation

As remarked in chapter 1 the planar replicator system does not admit stable limit cycles and hence generic Hopf bifrucations which give rise to such limit cycles are excluded. However, a degenerate type of Hopf bifurcation does occur. The canonical form of this degenerate type is given by the system $\dot{r} = \theta r$, $\dot{\psi} = 1$, where (r,ψ) are polar coordinates and θ is the real parameter. The phase portraits are sketched in Figure 3.1.

$\theta < 0$ $\theta = 0$ $\theta > 0$

Figure 3.1

Origin is a repellor for $\theta > 0$ and an attractor for $\theta < 0$. At $\theta = 0$ it is a centre and orbits are the circles $r =$ constant. As θ goes through zero the repellor is turned into an attractor, but unlike the generic case no limit cycle is born. The vector field at $\theta = 0$ is Hamiltonian and is highly degenerate; in fact it has infinite codimension in the space of all vector fields in the plane [4]. However such bifurcations arise in Hamiltonian dynamical systems or systems exhibiting a particular group of symmetry.

In our system, degenerate Hopf bifurcations occur in C_1 and C_7. They were studied by Zeeman using Lyapunov functions with which the global bifurcations involved can also be understood. The results are in the following two propositions which are proved in [25] for the class of central matrices. We have refined the proof in order to remove this restriction.

Proposition 3.1

A transversal deformation of a cod 1 matrix in $\{\det = 0\}$ in C_1 induces a degenerate Hopf bifurcation.

Proof

Up to equivalence we can work in the sign class S_1. Let (U,f) be a representative unfolding of the deformation in question with $f(\varepsilon) \in S_1$, $\forall\, \varepsilon \in U$. Then for each $\varepsilon \in U$, $\mathrm{adj}(f(\varepsilon))$ has positive entries and hence, by proposition 1.3 (ix), $\Delta_{f(\varepsilon)}$ has a unique fixed point in $\mathring{\Delta}$. Using the remark after corollary 1.6 we can now centralize the unfolding (U,f) for $\forall\ \varepsilon \in U$ and hence assume that, up to equivalence, f has the form:

$$f(\varepsilon) = \begin{pmatrix} 0 & \theta(\varepsilon) + a_1(\varepsilon) & \theta(\varepsilon) - a_1(\varepsilon) \\ \theta(\varepsilon)-a_2(\varepsilon) & 0 & \theta(\varepsilon) + a_2(\varepsilon) \\ \theta(\varepsilon) + a_3(\varepsilon) & \theta(\varepsilon) - a_3(\varepsilon) & 0 \end{pmatrix} \text{with } 0 \le |\theta(\varepsilon)| < a_i(\varepsilon)\ ,$$

$i = 1,2,3$. Then $\det f = 2\theta(\theta^2+\rho)$, where $\rho = \sum_{i<j} a_i a_j > 0$. (Here and hereafter arguments will be suppressed whenever convenient).
$\det f(0) = 0 \Rightarrow \theta(0) = 0$ and the transversality condition becomes $\left.\frac{d\theta}{d\varepsilon}\right|_{\varepsilon=0} \neq 0$ which enables us, by reducing U if necessary, to reparametrize the unfolding and take θ as the parameter. We shall denote this reparametrized equivalent unfolding by (U,f) again, where f is as above except that θ is now the independent parameter.

Construct a Lyapunov function V_θ in $\mathring{\Delta}$ as follows. Let $b_i(\theta) = \frac{b(\theta)}{a_i(\theta)}$, $i = 1,2,3$, where $b(\theta) = (\Sigma \frac{1}{a_i(\theta)})^{-1}$. Then $b_i(\theta) > 0$ and $\sum_i b_i(\theta) = 1$. For $\theta \in U$, define $V_\theta : \mathring{\Delta} \to \mathbb{R}$ by

$$V_\theta(x) = \frac{\prod_i x_i^{b_i(\theta)}}{\sum_i b_i(\theta) x_i}\ .$$

$V_\theta(x) > 0$, $\forall\ x \in \mathring{\Delta}$ and $V_\theta(x) \to 0$ as $x \to \partial\Delta$. V_θ takes the maximum value of one at the barycentre E. Level curves of V_θ are closed curves surrounding E.

Differentiating V_θ along the orbits of $\Delta_{f(\varepsilon)}$ we obtain after some work (see [25]):

$$V_\theta(x) = \theta W_\theta(x) \text{ where } W_\theta(x) = \frac{\prod_i x_i^{b_i(\theta)} \sum_{i<j} b_i(\theta) b_j(\theta)(x_i - x_j)^2}{(\Sigma\, b_i(\theta) x_i)^2} > 0 \quad .$$

If follows that, at $\theta = 0$, orbits of $\dot{\Delta}_{f(0)}$ are closed curves of $V_0(x) =$ constant, which fill out $\overset{o}{\Delta}$. When $\theta > 0$ all orbits of $\Delta_{f(\theta)}$ intersect the level curves $V_\theta(x) =$ constant transversally and hence spiral towards the maximum of V_θ at E. Finally, when $\theta < 0$, the reverse situation occurs and all orbits in $\overset{o}{\Delta}\backslash E$ spiral outwards from E. (Figure 3.2)

Figure 3.2

Therefore a degenerate Hopf bifurcation is induced at E similar to the canonical form of Figure 3.1. □

The other case is much more involved.

<u>Proposition 3.2</u>

A transversal deformation of a cod 1 matrix in $\{\det = 0\}$ separating (7_1) and (-7_1) in C_7 induces a degenerate Hopf bifurcation.

<u>Proof</u>

We will work in the sign class S_7 and, by steps similar to those in the proof of proposition 3.1, we can assume that a representative unfolding (U,f) of the deformation has the form

$$f(\theta) = \begin{pmatrix} 0 & \theta + a_1(\theta) & \theta - a_1(\theta) \\ \theta - a_2(\theta) & 0 & \theta + a_2(\ ) \\ \theta + a_3(\ ) & \theta - a_3(\theta) & 0 \end{pmatrix} \text{ with } a_1(\theta), a_2(\theta) > 0 ,$$

$a_3(0) < 0$, $|\theta| < |a_i(\theta)|$ and $\beta(\theta) = \sum_{i<j} a_i(\theta) a_j(\theta) > 0$ for $\theta \in U$.

Note that in this central form $\beta_i(\theta) = \theta^2 + \rho(\theta)$ and hence the conditions above are equivalent to those in theorem 1.9. For $\forall\, \theta \in U$, X_1 is an attractor, X_2 a repellor, X_3 a saddle and E a fixed point. There is a saddle

$$M_\theta = \left(\frac{a_1(\theta)-\theta}{a_1(\theta)-a_3(\theta)-2\theta} \ , \ 0 \ , \ \frac{-a_3(\theta)-\theta}{a_1(\theta)-a_3(\theta)-2\theta}\right) \text{ on } \overset{\circ}{\overline{X_1X_3}} \text{ and another}$$

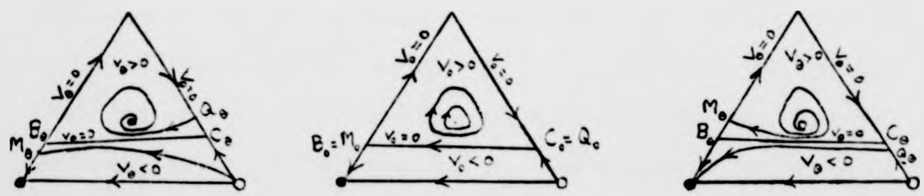
$$Q_\theta = \left(0 \ , \ \frac{a_2(\theta)+\theta}{a_2(\theta)-a_3(\theta)+2\theta} \ , \ \frac{-a_3(\theta)+\theta}{a_2(\theta)-a_3(\theta)+2\theta}\right) \text{ on } \overset{\circ}{\overline{X_2X_3}} \ .$$

Let $V_\theta : \overset{\circ}{\Delta} \to \mathbb{R}$ be defined by $V_\theta(x) = (\prod_i x_i^{-b_i(\theta)})\Sigma\, b_i(\theta)x_i$ where $b_i = \frac{b}{a_i}$ and $b = (\sum_i \frac{1}{a_i})^{-1}$ are as in proposition 3.1. Here $b_3 > 0$ but $b, b_1, b_2 < 0$. V_θ takes the maximum value of one at E and vanishes on the line $\overline{B_\theta C_\theta}$, where

$$B_\theta = \left(\frac{a_1(\theta)}{a_1(\theta)-a_3(\theta)}, 0, \frac{-a_3(\theta)}{a_1(\theta)-a_3(\theta)}\right) \text{ and } C_\theta = \left(0, \frac{a_2(\theta)}{a_2(\theta)-a_3(\theta)}, \frac{-a_3(\theta)}{a_2(\theta)-a_3(\theta)}\right),$$

and on the half open lines $\overline{X_1X_3}\backslash\{X_1\}$ and $\overline{X_2X_3}\backslash\{X_2\}$.

The positive level curves of V_θ are closed and fill the interior of the triangle $X_\theta B_\theta C_\theta$ while the negative level curves join X_1 and X_2 filling the interior of the trapezium $X_1X_2C_\theta B_\theta$. Furthermore $V_\theta(x) \to -\infty$ as $X \to$ interior of $\overline{X_1X_2}$. (Figure 3.3)



$\theta > 0$ $\qquad\qquad$ $\theta = 0$ $\qquad\qquad$ $\theta < 0$

Figure 3.3

Differentiating V_θ along orbits of $\Delta_{f(\theta)}$ we obtain

$$V_\theta(x) = \theta W_\theta(x) \quad \text{where} \quad W_\theta(x) = -(\prod_i x_i^{-b_i(\theta)}) \sum_i \sum_j b_i(\theta)b_j(\theta)(x_i-x_j)^2 .$$

A lemma in [25] shows that $W_\theta(x) > 0$ for $x \in \overset{\circ}{\Delta}\backslash E$. When $\theta = 0$,

$V_0(x) = 0$; hence orbits of $\Delta_{f(0)}$ in $\mathring{\Delta}$ are level curves of V_0 . Note that $B_0 = M_0$ and $C_0 = Q_0$ and hence the open line $\overline{Q_0M_0}$ is an orbit which is therefore a saddle connection. Above this line orbits are closed and fill the interior of the triangle $X_3Q_0M_0$. In the trapezium $X_2Q_0M_0X_1$ orbits flow from the repellor X_2 to the attractor X_1 . When $\theta > 0$, $V_\theta > 0$ and orbits of $\Delta_{f(\theta)}$ intersect level curves of V_θ transversally in $\mathring{\Delta}\backslash E$. We observe that $\rho = \sum_{i<j} a_i a_j > 0 \Rightarrow (a_1+a_3)a_2 > -a_1a_3 \Rightarrow a_1+a_3 > 0$ and similarly $a_2 + a_3 > 0$. So it follows that

$$\frac{-a_3(\theta)-\theta}{a_1(\theta)-a_3(\theta)-2\theta} < \frac{-a_3(\theta)}{a_1(\theta)-a_3(\theta)} \quad \text{and} \quad \frac{-a_3(\theta)+\theta}{a_2(\theta)-a_3(\theta)+2\theta} > \frac{-a_3(\theta)}{a_2(\theta)-a_3(\theta)}$$

and hence M_θ lies below B_θ while Q_θ is above C_θ . Therefore the outset of Q_θ cannot cross $\overline{B_\theta C_\theta}$ on which $V_\theta = 0$ and must go to E . The inset of M_θ must therefore come from X_2 ; it separates the basins of attraction of E and X_1 . For $\theta < 0$, the reverse situation occurs: The inset of M_θ must come from the repellor E , while the outset of Q_θ goes to X_1 and separates the basins of repulsion of E and X_2 .

Therefore a degenerate Hopf bifurcation is induced at E and at the same time a crossing of the inset of M_θ and the outset of Q_θ through the saddle connection Q_0M_0 takes place. □

The local and global behaviour of these bifurcations is therefore understood. We shall return to them again in chapter 6.

3.3 Exchange of Stability Bifurcation

In this section we prove that transversal deformations of cod 1 matrices in $\{a_{ij} = 0\}$ and $\{\beta_i = 0\}$ induce exchange of stability (transcritical) bifurcation, when the flow is extended to a neighbourhood of Δ in the invariant plane $\sum_i x_i = 1$. The canonical form of this bifurcation is given by the system

$$\text{(a)} \quad \begin{cases} \dot{x} = g(x,\varepsilon) = \pm x^2 \pm \varepsilon x \\ \dot{\varepsilon} = 0 \end{cases} \qquad x,\varepsilon \in \mathbb{R} .$$

Figure 3.4 gives the phase portrait when the plus signs are taken in both terms. For the flow restricted to the invariant lines ε = Constant < 0 , the point $x = 0$ is an attractor and $x = -\varepsilon$ is a repellor; whereas for the invariant lines ε = Constant > 0 , $x = 0$ is a repellor and $x = -\varepsilon$ an attractor. Therefore as ε goes through zero the fixed points cross each other and exchange stability.

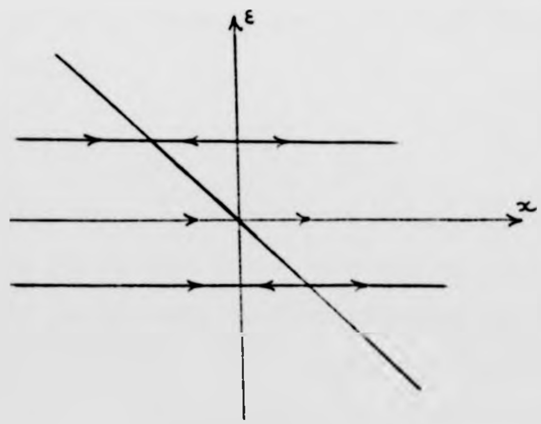


Figure 3.4

In general consider the smooth vector field

(b) $$\begin{cases} \dot{x} = f(x,\varepsilon) \\ \dot{\varepsilon} = 0 \end{cases} \qquad x,\varepsilon \in \mathbb{R}$$

satisfying the conditions

b(i) $$f(0,0) = \frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial \varepsilon}(0,0)$$

b(ii) $$\frac{\partial^2 f}{\partial x \partial \varepsilon}(0,0) \neq 0$$

b(iii) $$\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0$$

Then it can be shown that (b) is locally equivalent to (a).

Conditions b(i)-b(iii) can be generalized for higher dimensions [12], but then they will be tedious and non-intuitive. It is by far better to use the celebrated <u>reduction theorem</u> [4] [12] for reducing the vector field to the centre manifold, which we will now explain. Let

(c) $$\dot{x} = h(x) , \qquad x \in \mathbb{R}^n$$

be a smooth family of vector fields with origin as an equilibrium point. Suppose that the linear part of h at origin has n_+, n_- and n_0 eigenvalues with respectively positive, negative and zero real parts. Let E_0 denote the subspace of $\mathbb{R}^n$ spanned by eigenvectors of eigenvalues with vanishing real part. Then by the centre manifold theorem

[12] there exists for each $r > 0$ a C^r-invariant manifold tangent to E_0 at origin. The reduction theorem now states that (c) is locally equivalent to the family:

$$\begin{cases} \dot{p} = h_0(p) & p \in \mathbb{R}^{n_0} \\ \dot{q} = -q & q \in \mathbb{R}^{n_-} \\ \dot{r} = r & r \in \mathbb{R}^{n_+} \end{cases}$$

where the first equation gives the dynamics on the centre manifold. In other words we can neglect eigenvalues of non-zero real part and obtain the topological picture by "suspending" the flow induced on the centre manifold.

Combining the above results, we deduce that if (c) has a two dimensional centre manifold on which the dynamics satisfies the conditions b(i)-b(iii), then (c) is locally equivalent to the system

$$\text{(d)} \quad \begin{cases} \dot{x} = \pm x^2 \pm \varepsilon x & x, \varepsilon \in \mathbb{R} \\ \dot{\varepsilon} = 0 & \\ \dot{q} = -\dot{q} & q \in \mathbb{R}^{n_-} \\ \dot{r} = r & r \in \mathbb{R}^{n_+} \end{cases}$$

which represents exchange of stability in higher dimensions. Taking plus signs in the first equation, $n_- = 1$ and $n_+ = 0$, the phase portraits of the flow on three invariant planes $\varepsilon < 0$, $\varepsilon = 0$ and $\varepsilon > 0$ are sketched in Figure 3.5.

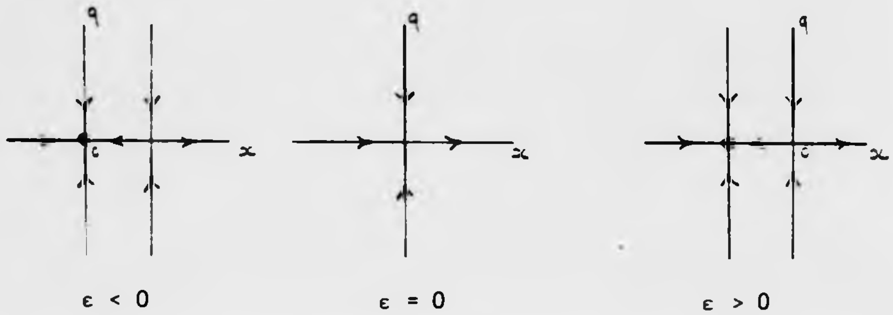


Figure 3.5

In the space of one parameter family of vector fields, families like (d) which undergo exchange of stability (ε is now considered as the parameter rather than a state variable) are not generic. In fact the only generic families are those undergoing Hopf bifurcation or saddle-node bifurcation [5]. However exchange of stability can arise when restrictions are present as in the replicator system.

We now begin to study exchange of stability at a vertex of Δ . <u>In the rest of this chapter we consider the replicator equations $\dot{x} = V^A(x)$ in a neighbourhood of Δ in the invariant plane $\sum_i x_i = 1$.</u>

<u>Proposition 3.3</u>

A transversal deformation of a cod 1 matrix $A \in \{a_{ij} = 0\}$ induces an exchange of stability at a vertex of Δ .

Proof

By a permutation of indices we can assume $A \in \{a_{12} = 0\}$. Let $(0,f)$ be a transversal deformation of A with

$$f(\varepsilon) = \begin{pmatrix} 0 & a_{12}(\varepsilon) & a_{13}(\varepsilon) \\ a_{21}(\varepsilon) & 0 & a_{23}(\varepsilon) \\ a_{31}(\varepsilon) & a_{32}(\varepsilon) & 0 \end{pmatrix}$$

where $a_{21}(0) = 0$ and, by transversality, $a_{21}'(0) \neq 0$. Reparametrizing and taking a_{12} as the new parameter and using coordinates (x_2,x_3) around X_1 we obtain the following autonomous system in (x_2,a_{21},x_3) :

$$(e) \begin{cases} \dot{x}_2 = x_2[a_{21}-a_{12}(0)x_2 + (a_{23}(0)-a_{13}(0)-a_{31}(0))x_3 + h.0.t] \\ \dot{a}_{21} = 0 \\ \dot{x}_3 = x_3[a_{31}(0) + h.0.t.] \end{cases}$$

where h.0.t. indicates higher order terms in x_2,x_3 and a_{21} . As $a_{31}(0) \neq 0$, the plane $x_3 = 0$ is the centre manifold at $(0,0,0)$, on which the dynamics is given by:

$$\begin{cases} \dot{x}_2 = a_{21}x_2 - a_{12}(0)x_2^2 + h.0.t. \\ \dot{a}_{21} = 0 \end{cases}$$

It is easily seen that the conditions b(i)-b(iii) are satisfied as $a_{12}(0) \neq 0$. Hence the deformation induces an exchange of stability bifurcation on $x_3 = 0$ whose suspension in the hyperbolic direction $\overline{X_1X_3}$ gives the overall bifurcation (see Figure 3.5 where both $a_{31}(0)$ and $a_{12}(0)$ are assumed positive). □

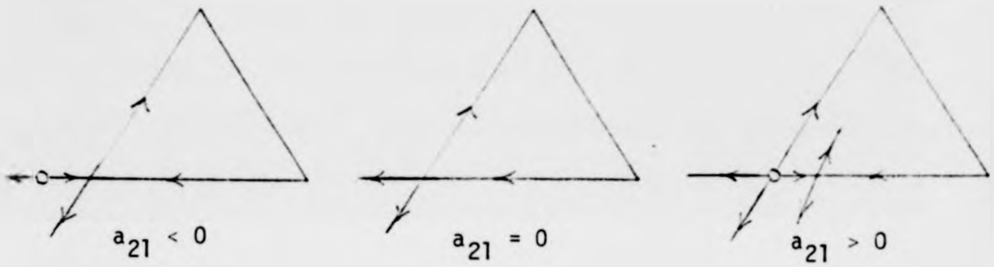


Figure 3.5

To deduce a similar result for the bifurcation at an interior point of an edge we need a lemma.

Lemma 3.4

If $A \in \{\beta_i = 0\}$ is a cod 1 matrix then $\det A \neq 0$.

Proof

We can write $A = W(\alpha).r$ with $\alpha \in Q$ and $r \in \mathbb{R}^{3^+}$ (see proposition 2.3). Assume WLG that $A \in \{\beta_3 = 0\}$; then $\alpha_1-\alpha_2 + \frac{\pi}{3} = 0$ and $\det W(\alpha) = \operatorname{Sin}\alpha_1\operatorname{Sin}\alpha_2\operatorname{Sin}\alpha_3$ $+ \operatorname{Sin}(\alpha_1- \frac{\pi}{3})\operatorname{Sin}(\alpha_2- \frac{\pi}{3})\operatorname{Sin}(\alpha_3- \frac{\pi}{3}) = \operatorname{Sin}(\alpha_2- \frac{\pi}{3})\operatorname{Sin}\alpha_2\operatorname{Sin}\alpha_3$ $+ \operatorname{Sin}(\alpha_2- \frac{2\pi}{3})\operatorname{Sin}(\alpha_2- \frac{\pi}{3})\operatorname{Sin}(\alpha_3- \frac{\pi}{3}) = \frac{1}{2} \operatorname{Sin}(\alpha_2- \frac{\pi}{3}) \operatorname{Cos}(\alpha_2-\alpha_3)$ $-\operatorname{Cos}(\alpha_2-\alpha_3- \frac{\pi}{3}) = \operatorname{Sin}\alpha_1\operatorname{Cos} \frac{\pi}{6} \operatorname{Cos}(\alpha_2-\alpha_3- \frac{\pi}{6}) =$ $\operatorname{Cos} \frac{\pi}{6} \operatorname{Sin}\alpha_1\operatorname{Sin}(\alpha_2-\alpha_3 + \frac{\pi}{3}) = \beta_1 \operatorname{Sin}\alpha_1 \neq 0$ since $W(\alpha)$ is a cod 1 matrix with $\beta_3 = 0$. □

Proposition 3.5

A transversal deformation of a cod 1 matrix $A \in \{\beta_i = 0\}$ induces an exchange of stability bifurcation at an interior point of $\overline{X_{i+1}X_{i+2}}$.

Proof

Assume $A \in \{\beta_3 = 0\}$ and let (U,f) be a transversal unfolding of A . Using lemma 1.5 we can further assume by multiplication with a suitable matrix and reversing time if necessary that:

$$f(\varepsilon) = \begin{pmatrix} 0 & a & a_{13}(\varepsilon) \\ a & 0 & a_{23}(\varepsilon) \\ a_{31}(\varepsilon) & a_{32}(\varepsilon) & 0 \end{pmatrix} \qquad \text{where} \quad \varepsilon \in U$$

and a is a positive constant. The point $H = (\frac{1}{2}, \frac{1}{2}, 0)$, the mid point of $\overline{X_1X_2}$, is then fixed for $\forall\ \varepsilon \in U$. Taking coordinates (x_3, x_1) , the vector field around H becomes (arguments are suppressed):

$$\text{(g)} \quad \begin{aligned} \dot{x}_3 &= {}_3[\frac{a_{32}+a_{31}-a}{2} + (a_{31}-a_{32})x_1 + \frac{(a_{31}-a_{32}-a_{13}-a_{23})}{2}x_3] + \text{h.O.t.} \\ \dot{x}_1 &= -\frac{a}{2}x_1 + \frac{(a_{13}-a_{23}-a_{31}-a_{32})}{4}x_3 + \text{h.O.t.} \end{aligned}$$

Note that $\beta_3 = a(a_{32}+a_{31}-a)$ and by transversality $\beta_3'(0) \neq 0$, so that we can take β_3 as the new parameter and assume $\beta_3 \in U_1$ say.

Furthermore, we can take the eigenvectors of the linear part of the vector field (g) at $\beta_3 = 0$ as new axes and put

$$\begin{cases} x_3 = \frac{a}{2} w \\ x_1 = v + (a_{13}(0)-a_{23}(0)-a_{31}(0)-a_{32}(0)) \frac{w}{4} \end{cases}$$

Then (g) will be reduced to the autonomous system:

$$\begin{cases} \dot{w} = \frac{1}{2a} \beta_3 w + (a_{31}(0)-a_{32}(0))vw - \frac{1}{2a} (\det f(0))w^2 + \text{h.O.t.} \\ \dot{\beta}_3 = 0 \\ \dot{v} = -\frac{a}{2} v + \text{h.O.t.} \end{cases}$$

The v direction (corresponding to x_1 direction) is hyperbolic for $\forall\ \beta_3 \in U_1$. Therefore as in [12] we seek a centre manifold $v = h(v,\beta_3)$ tangent to (w,β_3) plane at $(w,\beta_3,v) = (0,0,0)$. We therefore write

$$v = h(w,\beta_3) = pw^2 + qw\beta_3 + r\beta_3^2 + \text{h.O.t.}$$

where p,q and r are constants that can be determined by using the invariance of the centre manifold. But this is not necessary. Since the contribution of h to the first term in (g) gives terms of order three in w and β_3, we deduce that up to second order terms the dynamics on the centre manifold is determined by

$$\begin{cases} \dot{w} = \frac{1}{2a} \beta_3 w - \frac{1}{2a} (\det f(0))\, w^2 + \text{h.O.t.} \\ \dot{\beta}_3 = 0 \end{cases}$$

Furthermore, we can take the eigenvectors of the linear part of the vector field (g) at $\beta_3 = 0$ as new axes and put

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$$\begin{cases} \dot{w} = \frac{1}{2a} \beta_3 w - \frac{1}{2a} (\det f(0))\, w^2 + \text{h.O.t.} \\ \dot{\beta}_3 = 0 \end{cases}$$

Since, by the lemma, $\det f(0) \neq 0$ we find that the conditions b(i)-b(iii) are satisfied and hence the vector field is the suspension of a system undergoing exchange of stability bifurcation (see Figure 3.6 where we have taken $\det f(0) > 0$).

$\beta_3 < 0$ $\beta_3 = 0$ $\beta_3 > 0$

Figure 3.6

Next we prove a technical lemma which we need in later chapters. Let (U,f) be an unfolding of the cod 1 matrix $f(0) \in \{a_{ij} = 0\}$ or $f(0) \in \{\beta_i = 0\}$.

Lemma 3.5

Given $\delta > 0$ there exists a neighbourhood N of the non-hyperbolic fixed point of $\Delta_{f(0)}$ in Δ such that the length of arcs of orbits of $\Delta_{f(\varepsilon)}$ in N is smaller than δ for $\forall \varepsilon \in U$.

Proof

(i) Consider first the case $f(0) \in \{a_{ij} = 0\}$. Assume WLG that the vector field is given by (e). By reversing time if necessary we can further assume that $a_{12}(0) > 0$. For $0 < \ell << 1$, put

$N_\ell = \{(x_2 . x_3) \mid |x_2| < \ell \, , \, |x_3| < \ell\}$. Suppose $a_{31}(0) > 0$, so that the phase portraits are as in Figure 3.5. $\dot{x}_3$ is positive, respectively negative, for points in N with positive, respectively negative, x_2 coordinate. Also $\frac{dx_3}{dx_2} = 1$ on a cubic curve C^+ which passes through the two fixed points and is close to the hyperbola $a_{31}(0)x_3 = x_2[\varepsilon - a_{12}(0)x_2 + (a_{23}(0) - a_{13}(0) - a_{31}(0))x_3]$ in N_ℓ for small ℓ . Similarly $\frac{dx_3}{dx_2} = -1$ on a cubic curve C^- which passes through the two fixed points and is close to the hyperbola
$a_{31}(0)x_3 = -x_2[\varepsilon - a_{12}(0)x_2 + (a_{23}(0) - a_{13}(0) - a_{31}(0))x_3]$ in N_ℓ for small ℓ.
For each $\varepsilon \in U$, these two curves partition N_ℓ into five regions (three regions when $\varepsilon = 0$) in each of which the vector field (e) satisfies either $|\frac{dx_3}{dx_2}| < 1$ or $|\frac{dx_3}{dx_2}| > 1$. (Figure 3.7)

<u>Figure 3.7</u>

An orbit of $\Delta_{f(\varepsilon)}$ in N_ℓ can only intersect at most once one of these curves. This is because a second intersection would require a change in sign of $\dot{x}_3$ which is not possible. But the length of an orbit in N_ℓ in a region where $\left|\frac{dx_3}{dx_2}\right| < 1$ is at most $\int_{-\ell}^{\ell} \sqrt{1 + (\frac{dx_3}{dx_2})^2}\, dx_2$ which is therefore less than $2\sqrt{2}\,\ell$. Similarly the length of an orbit in N_ℓ in a region where $\left|\frac{dx_3}{dx_2}\right| > 1$ is at most $\int_{-\ell}^{\ell} \sqrt{1 + (\frac{dx_2}{dx_3})^2}\, dx_3$ which is again less than $2\sqrt{2}\,\ell$. We conclude that the total length of an orbit in N_ℓ is at most $4\sqrt{2}\ell$ and the result follows. Same argument holds when $a_{31}(0) < 0$.

(ii) When $f(0) \in \{\beta_i = 0\}$ we use the vector field (g) and the proof is exactly as in (i). □

Remark 3.7

The property in the above lemma always holds for a hyperbolic fixed point of a C^1-family of planar vector fields. More precisely if $\dot{x} = X(x,\varepsilon)$ is a family of vector fields with $X: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ a C^1-map and $0 \in \mathbb{R}^2$ a hyperbolic fixed point of $\dot{x} = X(x,0)$, then given $\delta > 0$ there exists a neighbourhood of $0 \in \mathbb{R}^2$ in which the length of orbits of $\dot{x} = X(x,\varepsilon)$ are smaller than δ for all ε with $|\varepsilon| < \varepsilon_0$ where ε_0 is a positive number. This is because the family is locally C^1-equivalent to its linear part [6] [14] and the desired property clearly holds for the linear family and hence, as it is preserved under a C^1-change of coordinates, for the original family as well. However this property is not in general true for a nonhyperbolic fixed point. For example the orbits of the flow

$$\begin{cases} \dot{r} = -r^k & k \geq 2 \\ \dot{\theta} = 1 & \end{cases}$$

have infinite length in any neighbourhood of origin as can easily be checked directly

$$\left(\frac{dr}{d\theta} = -r^k \Rightarrow L = \int ds = \int_0^{r_0} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr = \right.$$

$$\left. \int_0^{r_0} \sqrt{1 + r^{2-2k}}\, dr > \int_0^{r_0} r^{1-k}\, dr = \infty \right) \quad . \qquad \square$$

Finally we note that the study of the cod 1 bifurcations in this chapter now enables us to completely determine the phase portraits of the corresponding cod 1 flows (see appendix 1).

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$$\int_0^{r_0} \sqrt{1 + r^{2-2k}}\, dr > \int_0^{r_0} r^{1-k}\, dr = \infty) \quad . \qquad \square$$

Finally we note that the study of the cod 1 bifurcations in this chapter now enables us to completely determine the phase portraits of the corresponding cod 1 flows (see appendix 1).

Chapter 4.

Versal deformations of cod 1 matrices (Part one)

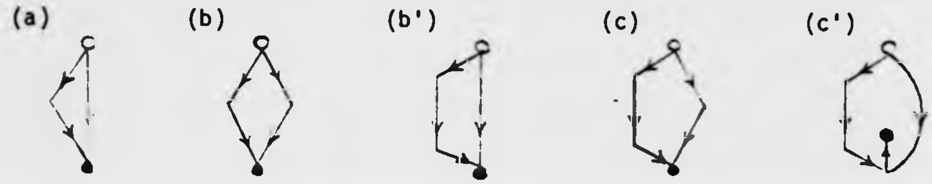
We will study topologically versal deformations of cod 1 matrices in two separate chapters. In the present we consider the cod 1 matrices satisfying the condition that no cycle of saddles appears in the flow induced by them or in the nearby stable flows. This is the case for matrices in I_i , $4 \le i \le 38$ (see appendix 1). The remaining three cases are much harder to deal with because of the presence of cycle of saddles. We will need some preliminary results in chapter 5 before we set to tackle these cases in chapter 6.

The results in this chapter were obtained by me independent from Peixoto's method in [19]. However in presenting them here I have modified my original version of proofs so that the reader familiar with that classic paper can follow the steps more easily.

4.1 Fundamental Domains

Throughout this chapter we consider the cod 1 matrices and the nearby stable matrices satisfying the above condition. Let A be such a matrix which can be stable or cod 1. Observe that all orbits in Δ_A go from a fixed point to a fixed point as there is no closed orbit or cycle of saddles. By lemma 3.5 and remark 3.6, the length of orbits in any neighbourhood of the fixed points of these flows is finite. It then follows by the compactness of Δ that all orbits in Δ have finite length. This property is essential for our constructions in this chapter.

Δ_A is partitioned into a finite number of closed domains each of which is bounded by an attractor, a repellor, a number of saddles and by saddle separatrices and orbits in $\partial\Delta$. We call these <u>fundamental domains</u> (F.D.) which are of five different types labelled as in Figure 4.1.



<u>Figure 4.1</u>

Consider transversal deformations of cod 1 matrices in I_i, $4 \le i \le 38$. At the bifurcation point a F.D. is transformed into a F.D. except that F.D's of type (a), (b) and (c) can also shrink to the closure of union of one, two or three orbits respectively which we denote by L^1, L^2 and L^3 (see Figure 4.4). The matrix of allowed transitions between F.D.'s is given by:

$$\begin{array}{l} (a) \\ (b) \\ (b') \\ (c) \\ (c') \end{array} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{array}{l} (a) \leftrightarrow L^1 \\ (b) \leftrightarrow L^2 \\ (c) \leftrightarrow L^3 \end{array}$$

We now seek to define coordinates (ϕ, z) for a point in a F.D. of the flow Δ_A. We start with defining ϕ. In each F.D. the attractor or the repellor, but not both, may undergo exchange of stability. If the repellor is not undergoing exchange of stability call it the base of the F.D., otherwise call the attractor the base. We note that in (c') the repellor never undergoes exchange of stability, so it is always the base. Pick a small fixed positive number r. Take the arc, C, of the circle of radius r with the base of the F.D. as its centre and parametrize it arcwise linearly from $\phi = 0$ to $\phi = 1$, anticlockwise if the repellor is the base, clockwise if the attractor is the base (Figure 4.2). Now define the ϕ coordinate of a point P in the interior of the F.D. to be the value of ϕ at which the orbit through P, $\sigma_A(P)$, intersects C. The points on the left and the right boundaries of the F.D. are given the ϕ coordinate 0 and 1 respectively. We consider type (c') as the limit case of Figure 4.3 for which ϕ is defined on both boundaries. However, in all cases, ϕ remains undefined for the repellor and the attractor points.

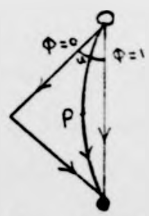


Figure 4.2

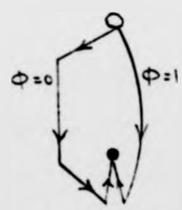


Figure 4.3

To define the z coordinate we first need to introduce what we call an R-curve. Given two points one on each side of the boundary of a F.D., an R-curve joining them is a curve which intersects the orbits of the F.D. with a definite ratio of arc length. We define it explicitly for type (a); it is similar in others. Let X and Y be opposite points on the boundary of a F.D. of type (a) with repellor A, attractor C and saddle B (Figure 4.4).

Put $\gamma = \dfrac{\ell(\overset{\frown}{AX})}{\ell(\overset{\frown}{AB}) + (\overset{\frown}{BC})}$ and

$$\lambda = \frac{\ell(\overset{\frown}{AY})}{\ell(\overset{\frown}{AY}) + \ell(\overset{\frown}{YC})}$$

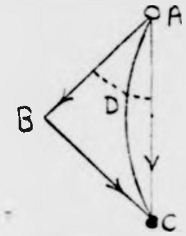


Figure 4.4

then the R-curve joining X and Y (always drawn by broken lines) is the curve which intersects the orbit with coordinate ϕ at a point D such that $\dfrac{\ell(\overset{\frown}{AD})}{\ell(\overset{\frown}{AD}) + \ell(\overset{\frown}{DC})} = (1 - \phi)\gamma + \phi\lambda$. We must now prove:

Lemma 4.1

R-curves are continuous.

Proof

As the proof is similar in F.D.'s of different type, we will give the proof only for Type (a). (Figure 4.4) All we need to check is that the length of orbits with $\phi \neq 0$ vary continuously with ϕ and that as $\phi \to 0$ the length of orbits tend to $\ell(\overset{\frown}{AB}) + \ell(\overset{\frown}{BC})$. To prove the first assertion take an orbit with coordinate $\phi_0 \neq 0$ and choose $\varepsilon > 0$. By lemma 3.6 and remark 3.7 there exists neighbourhoods of A and C such that the length of orbits in each is less than $\frac{\varepsilon}{3}$. Outside these neighbourhoods, the continuity of the differential orbits of the F.D. with a definite ratio of arc length. We define it explicitly for type (a); it is similar in others. Let X and Y be opposite points on the boundary of a F.D. of type (a) with repellor A, attractor C and saddle B (Figure 4.4).

Put $\gamma = \dfrac{\ell(\overset{\frown}{AX})}{\ell(\overset{\frown}{AB}) + (\overset{\frown}{BC})}$ and

$$\lambda = \frac{\ell(\overset{\frown}{AY})}{\ell(\overset{\frown}{AY}) + \ell(\overset{\frown}{YC})}$$

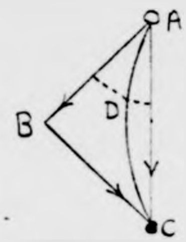


Figure 4.4

then the R-curve joining X and Y (always drawn by broken lines) is the curve which intersects the orbit with coordinate ϕ at a point D such that $\dfrac{\ell(\overset{\frown}{AD})}{\ell(\overset{\frown}{AD}) + \ell(\overset{\frown}{DC})} = (1 - \phi)\gamma + \phi\lambda$. We must now prove:

Lemma 4.1

R-curves are continuous.

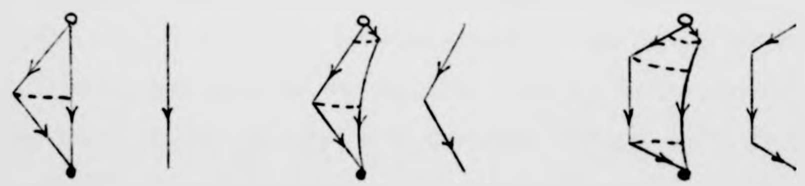
Proof

As the proof is similar in F.D.'s of different type, we will give the proof only for Type (a). (Figure 4.4) All we need to check is that the length of orbits with $\phi \neq 0$ vary continuously with ϕ and that as $\phi \to 0$ the length of orbits tend to $\ell(\overset{\frown}{AB}) + \ell(\overset{\frown}{BC})$. To prove the first assertion take an orbit with coordinate $\phi_0 \neq 0$ and choose $\varepsilon > 0$. By lemma 3.6 and remark 3.7 there exists neighbourhoods of A and C such that the length of orbits in each is less than $\frac{\varepsilon}{3}$. Outside these neighbourhoods, the continuity of the differential equations with respect to initial conditions [1] implies that the length of orbits with ϕ sufficiently close to ϕ_0 differs from that of ϕ_0 by at most $\frac{\varepsilon}{3}$. Hence the total length of orbits with ϕ close to ϕ_0 differs from that of ϕ_0 by at most ε and the assertion follows. The second assertion is proved similarly by considering, in addition to the two neighbourhoods around A and C , a neighbourhood around C in which length of orbits remain small. □

Let (U,f) be a transversal unfolding of $f(0) \in I_i$, $4 \le i \le 38$. We now proceed to partition the fundamental domains in $\Delta_{f(\varepsilon)}$, $\varepsilon \in U$, by R-curves as follows.

Step 1

Start with the F.D.'s of type (a), (b) and (c) which shrink to L^1 , L^2 and L^3 respectively. Through each saddle take the R-curve joining it to the point on the opposite side of the boundary which lengthwise divides the boundary of that side in the same ratio as the saddle divides its own side of the boundary. (Figure 4.4)



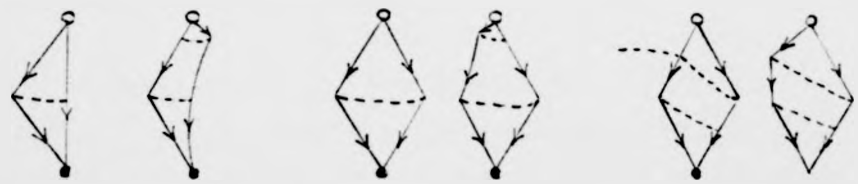
(a) <-> L^1 (b) <-> L^2 (c) <-> L^3

Figure 4.4

Step 2

We now take an R-curve through each saddle point in all other

F.D.'s : If the saddle is undergoing exchange of stability with the repellor (attractor) of the F.D., choose the opposite point to be at the same distance on the boundary to the repellor (attractor) as the saddle is to the repellor (attractor). In other cases, take the R-curve from the saddle to the point on the opposite side which is the endpoint of an R-curve determined by step 1 in a neighbouring F.D., <u>or</u>, if such a point does not exist, to the saddle on the opposite side, <u>or</u>, if neither of these points exist, to the point on the opposite side with the same dividing ratio of the boundary as in step 1. Figure 4.5 illustrates examples of these cases.



(a) <-> (b)　　(b) <-> (c)　　(b) <-> (c)

Figure 4.5

<u>Step 3</u>

Complete the partition of $\Delta_{f(\varepsilon)}$, $\varepsilon \in U$, by taking R-curves from the endpoints of the R-curve(s) determined in the two steps above to similar endpoint(s) on the opposite side or, in the absence of such endpoint(s), to the point(s) with the same dividing ratio of the boundary.

The complete result is given in appendix 1. The partition of $\Delta_{f(\varepsilon)}$, $\varepsilon \in U$, obtained in this way consists of a number of sub-domains in each F.D. ; a sub-domain is either triangular i.e. bounded by a repellor (an attractor), two semi-orbits flowing from the repellor (flowing to the attractor) and an R-curve, or rectangular i.e. bounded by two orbit segments and two R-curves. (Figure 4.6)

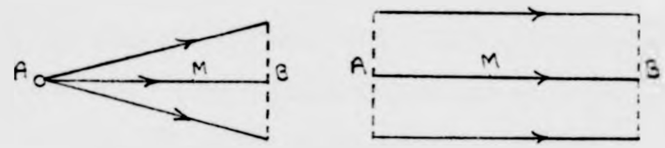


Figure 4.6

Now let M be a point on an orbit AB in a sub-domain of a F.D. with, say, A between M and the base of the F.D. We define the z coordinate of M to be $\frac{\ell(\widehat{AM})}{\ell(\widehat{AB})}$.

Therefore, given $\varepsilon \in U$, any point $M \in \Delta$ lies in a sub-domain of a F.D. in $\Delta_{f(\varepsilon)}$ and has accordingly a pair of (ϕ,z) coordinates with $0 \leq \phi \leq 1$ and $0 \leq z \leq 1$. Points which lie on the boundary of two or more sub-domains will have accordingly different coordinates with respect to the sub-domains in question. For simplicity we will not refer explicitly to any particular sub-domain and always assume that (ϕ,z) refers to the coordinates of M with respect to a given sub-domain of $\Delta_{f(\varepsilon)}$.

4.2 The Family of Homeomorphisms H_ε

Let $(0,f)$ and $(0,\bar{f})$ be transversal deformations of cod 1 matrices $f(0)$ and $\bar{f}(0)$ both in the same stratum I_i , $4 \leq i \leq 38$. We shall prove in this section that these two deformations are equivalent.

Let (U,f) and $(\bar{U},\bar{f})$ be two representative unfoldings. There exists $U^* \subset U \cap \bar{U}$ with the following properties:

(i) $f,\bar{f} : U^* \to Z_3$ are embeddings.

(ii) $f(\varepsilon)$ and $\bar{f}(\varepsilon)$ are stable for $\varepsilon \in U^*\backslash 0$.

We can assume $f(\varepsilon)$ and $\bar{f}(\varepsilon)$ belong to the same stable class for $\varepsilon \in U^* \cap \mathbb{R}^+$ (and consequently to the other stable class for $\varepsilon \in U^* \cap \mathbb{R}^-$) , for otherwise we can reparametrize $(U^*,\bar{f})$ by $\varepsilon \to -\varepsilon$. We now construct a family of homeomorphisms $H_\varepsilon : \Delta \to \Delta$ such that, for each $\varepsilon \in U^*$, H_ε induces an equivalence between $\Delta_{f(\varepsilon)}$ and $\Delta_{\bar{f}(\varepsilon)}$. Let x be a point with coordinates (ϕ,z) in a sub-domain of a F.D. in $\Delta_{f(\varepsilon)}$. There is a unique point x' with coordinates (ϕ,z) in the sub-domain of the F.D. in $\Delta_{\bar{f}(\varepsilon)}$ which corresponds to that of x in $\Delta_{f(\varepsilon)}$. We define x' to be the image of x under H_ε , and we prove:

<u>Proposition 4.2</u>

H_ε induces an equivalence between $\Delta_{f(\varepsilon)}$ and $\Delta_{\bar{f}(\varepsilon)}$ for each $\varepsilon \in U^*$.

<u>Proof</u>

Clearly, for each $\varepsilon \in U^*$, H_ε is a bijection which maps oriented orbits of $\Delta_{f(\varepsilon)}$ to those of $\Delta_{\bar{f}(\varepsilon)}$. To prove the continuity of H_ε and its inverse, fix $\varepsilon \in U^*$ and consider a sequence of points M_j and a point M in Δ with coordinates (ϕ_j,z_j) and (ϕ,z) with respect to a sub-domain of a F.D. in $\Delta_{f(\varepsilon)}$. Then by our construction of H_ε

the result will follow once we show that $M_j \to M$ iff $(\phi_j, z_j) \to (\phi, z)$. [When M is on the boundary of more than one sub-domain by the latter convergence we mean the convergence of the subsequence in each sub-domain to the coordinates of M in that sub-domain.] But this is like lemma 4.1 a straightforward consequence of the continuity of the solutions of the differential equations with respect to initial conditions, combined with lemma 3.6 and remark 3.7. □

To prove that H_ε depends continuously on ε we must first examine how the length of saddle separatrices vary with ε . For this we need the stable manifold theorem which we state below.

Let M be a smooth compact manifold. Define two smooth sub-manifolds S and S' of M to be δ-close ($\delta > 0$) if there exists a smooth diffeomorphism $h : S \to S' \subset M$ such that $i'h$ is δ-close to i in the C^∞-topology of the space of maps of M into itself, where $i:S \to M$ and $i':S' \to M$ denote the inclusions. Let Diff(M) be the space of all smooth diffeomorphisms of M into itself. Then we have:

<u>Proposition 4.3</u> (The Stable Manifold Theorem) [16] [18]

Let $f \in \mathrm{Diff}(M)$, P a hyperbolic fixed point of f and E^s the stable subspace of the linear space $(Df)_P$, then:

(i) $W^s(P)$ is a smooth injectively immersed manifold in M and the tangent space to $W^s(P)$ at P is E^s .

(ii) Let $D \subset W^s(P)$ be an embedded disc containing P. Consider a neighbourhood $N \subset \mathrm{Diff}(M)$ such that each $g \in N$ has a unique hyperbolic fixed point P_g contained in a certain neighbourhood of P. Then, given $\delta > 0$, there exists a neighbourhood $\bar{N} \subset N$ of f such that, for each $g \in \bar{N}$, there exists a disc $D_g \subset W^s(P_g)$ that is δ-close to D_f.

By the usual method of considering the time one map of a vector field $X \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the space of smooth vector fields on M, the above theorem can be shown to hold for vector fields as well [18].

Now we claim that the length of a saddle separatrix varies continuously with ε except at the bifurcation points inducing (b) <-> L^2 and (c) <-> L^3 (Figure 4.4), in which case we assert that the length of the saddle separatrix tends to the sum of lengths of the corresponding two or three orbits respectively. Consider first the general case. Fix $\varepsilon_0 \in U^*$ and pick $\delta > 0$. The inset (or outset) of the saddle in question, S_ε, flows from a repellor (or to an attractor) T_ε (Figure 4.7). By lemma 3.6 and remark 3.7 there exists a neighbourhood M of S_{ε_0} and a neighbourhood N of T_{ε_0} such that the length of orbits of $\Delta_{f(\varepsilon)}$ in M and N is smaller than $\frac{\delta}{3}$ for ε close to ε_0. The inset (or outset) of S_ε intersects the boundary of M at K_ε and that of N at L_ε. Now we note that $K_\varepsilon \to K_{\varepsilon_0}$ as $\varepsilon \to \varepsilon_0$. If S_{ε_0} is

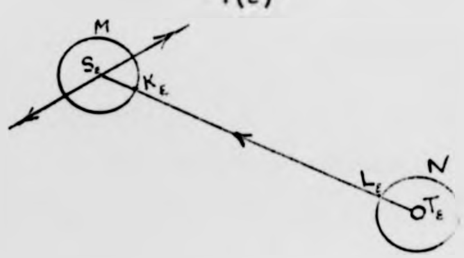


Figure 4.7

hyperbolic this follows from proposition 4.3(ii) (formulated for vector fields). When S_{ε_0} is non-hyperbolic, the family of vector fields is locally at S_{ε_0} equivalent to the canonical family of vector fields undergoing exchange of stability bifurcation (Figure 3.5) and it follows again that $K_\varepsilon \to K_{\varepsilon_0}$ as $\varepsilon \to \varepsilon_0$. But the continuity of the solutions of the differential equations with respect to the <u>initial conditions and the parameter</u> [2] implies that $\ell(\overset{\curvearrowright}{L_\varepsilon K_\varepsilon})$ differs from $\ell(\overset{\curvearrowright}{L_{\varepsilon_0} K_{\varepsilon_0}})$ by at most $\frac{\delta}{3}$ for ε close to ε_0 . Hence $\ell(\overset{\curvearrowright}{T_\varepsilon S_\varepsilon})$ differs from $\ell(\overset{\curvearrowright}{T_{\varepsilon_0} S_{\varepsilon_0}})$ by at most δ for ε sufficiently near ε_0 and our claim is proved. For the cases (b) <-> L^2 and (c) <-> L^3 we take additional neighbourhoods, around the other fixed points, in which the length of orbits remain small and the claim follows again. (Figure 4.8)

(b) <-> L^2 (c) <-> L^3

<u>Figure 4.8</u>

<u>Proposition 4.4</u>

H_ε depends continuously on $\varepsilon \in U^*$.

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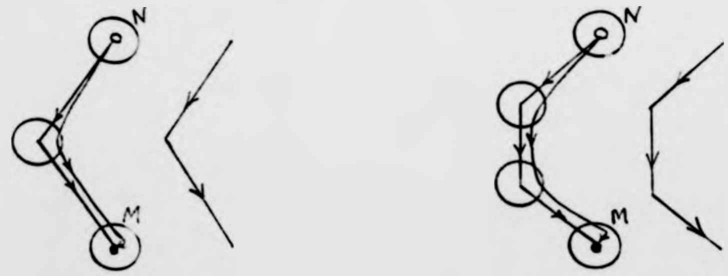


(b) <-> L^2 (c) <-> L^3

<u>Figure 4.8</u>

<u>Proposition 4.4</u>

H_ε depends continuously on $\varepsilon \in U^*$.

Proof

We shall prove that $H_\varepsilon(x) \to H_{\varepsilon_0}(x_0)$ as $(x,\varepsilon) \to (x_0,\varepsilon_0) \in \Delta \times U^*$. Assume x_0 belongs to a rectangular sub-domain of a F.D. in $\Delta_{f(\varepsilon_0)}$. The case of a triangular sub-domain is proved in a similar way. Let $\sigma_{f(\varepsilon)}(x)$, $\varepsilon \in U^*$, intersect the circle of radius r (defined in section 4.1) around the base of the F.D. of x at $M_\varepsilon(x)$ and intersect the R-curves, K_ε and L_ε, of the boundary of the rectangular sub-domain of x, at $K_\varepsilon(x)$ and $L_\varepsilon(x)$. For the flow $\Delta_{f(\varepsilon)}$, let the corresponding points be $\bar{M}_\varepsilon(x)$, $\bar{K}_\varepsilon(x)$ and $\bar{L}_\varepsilon(x)$ (Figure 4.9).

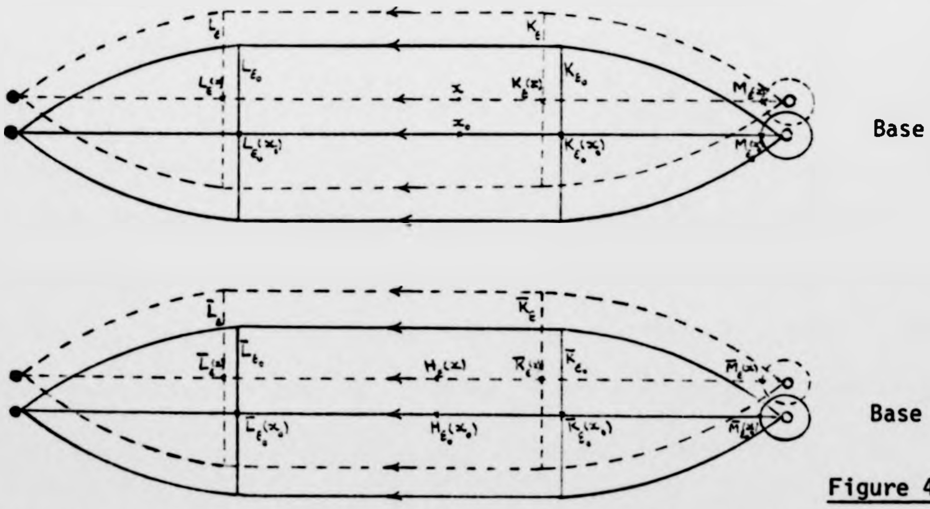


Figure 4.9

The coordinates of x with respect to its sub-domain in $\Delta_{f(\varepsilon)}$ is denoted as before by (ϕ,z). By the continuity of the solutions of the differential equations with respect to the initial conditions and the parameter we have $M_\varepsilon(x) \to M_{\varepsilon_0}(x_0)$ as $(x,\varepsilon) \to (x_0,\varepsilon_0)$. From this it follows that $u \to u_0$ as $(x,\varepsilon) \to (x_0,\varepsilon_0)$. Next we show that $z \to z_0$ as $(x,\varepsilon) \to (x_0,\varepsilon_0)$. The R-curve K_ε divides the orbit $\sigma_{\Delta_{f(\varepsilon)}}(x)$ lengthwise with ratio $(1-\phi)\gamma + \phi\lambda$ (see the construction of R-curves in section 4.1) where γ and λ depend continuously on the length of the boundaries of the sub-domains which in turn vary continuously with ε by the claim preceding this proposition. Hence as $\varepsilon \to \varepsilon_0$ the ratio $(1-\phi)\gamma + \phi\lambda$ tends to the ratio $(1-\phi_0)\gamma_0 + \phi_0\lambda_0$ with which K_{ε_0} divides the orbit $\sigma_{\Delta_{f(\varepsilon)}}(x_0)$. Furthermore by the continuity of solutions of the differential equations with respect to the initial conditions and the parameter, the length of the orbit $\sigma_{f(\varepsilon)}(x)$ tends to the length of the orbit $\sigma_{f(\varepsilon_0)}(x_0)$ as $(x,\varepsilon) \to (x_0,\varepsilon_0)$. Hence $K_\varepsilon(x) \to K_{\varepsilon_0}(x_0)$ and similarly $L_\varepsilon(x) \to L_{\varepsilon_0}(x_0)$ as $(x,\varepsilon) \to (x_0,\varepsilon_0)$. It then follows that $z \to z_0$ as $(x,\varepsilon) \to (x_0,\varepsilon_0)$. Now we reverse the argument in $\Delta_{\bar{f}(\varepsilon)}$. Since $\phi \to \phi_0$ as $(x,\varepsilon) \to (x_0,\varepsilon_0)$ we have $\bar{M}_\varepsilon(x) \to \bar{M}_{\varepsilon_0}(x_0)$ as $(x,\varepsilon) \to (x_0,\varepsilon_0)$. Furthermore $\bar{R}_\varepsilon$ divides the orbit $\sigma_{\Delta_{f(\varepsilon)}}(H_\varepsilon(x))$ lengthwise with ratio $(1-\phi)\bar{\gamma} + \phi\bar{\lambda}$ where $\bar{\gamma}$ and $\bar{\lambda}$ vary continuously with ε . Hence $\bar{R}_\varepsilon(x) \to R_{\varepsilon_0}(x_0)$ and similarly $\bar{L}_\varepsilon(x) \to \bar{L}_{\varepsilon_0}(x_0)$ as $(x,\varepsilon) \to (x_0,\varepsilon_0)$. Finally, since $z \to z_0$ as $(x,\varepsilon) \to (x_0,\varepsilon_0)$, we obtain $H_\varepsilon(x) \to H_{\varepsilon_0}(x_0)$ as $(x,\varepsilon) \to (x_0,\varepsilon_0)$. [Note that when x_0 belongs to the boundary of more than one sub-domain we get the same result by considering the subsequences of (x,ε) in each of the corresponding sub-domains of $\Delta_{f(\varepsilon)}$]. □

We have therefore proved the following result.

Theorem 4.5

Any two transversal deformations of the cod 1 matrices in I_i $(4 \leq i \leq 38)$ are equivalent. □

Corollary 4.6

Any transversal deformation of a cod 1 matrix in I_i $(4 \leq i \leq 38)$ is topologically miniversal and all these matrices have "codimension" one.

Proof

Let $(0,g)_k$ be a k-deformation of $A \in I_i$ and $(0,f)$ a transversal deformation of A. We want to prove that the deformation $(0,g)_k$ is equivalent to one induced from $(0,f)$. Let U^* be as in the beginning of this section i.e. $f : U^* \to Z_3$ is an embedding and $f(\varepsilon)$ is stable for $\varepsilon \in U^*\backslash 0$. Let $(U,g)_k$ be a representative k-unfolding of $(0,g)_k$. Consider the foliation of Z_3 whose leaves are given by $\{B \in Z_3 \mid \beta_i(B) = t , t \in \mathbb{R}\}$ if $\beta_i(A) = 0$ and by $\{B \in Z_3 \mid (B)_{ij} = t , t \in \mathbb{R}\}$ if $(A)_{ij} = 0$. By making U and U^* smaller if necessary we can assume the projection map, Π, of the foliation to Im f to be a single-valued continuous function. Now let $j : U \to \mathbb{R}$ be given by $j = f^{-1} \circ \Pi \circ g$. Then $(0,\bar{f})_k$ with $\bar{f} = f \circ j$ is a k-deformation of A which is induced from $(0,f)$. Furthermore for $\varepsilon \in U$, H_ε constructed as in the proof of theorem 4.5 gives an equivalence between $g(\varepsilon)$ and $f(\varepsilon)$ which varies continuously with ε. Hence $(0,g)_k$ is equivalent to $(0,\bar{f})_k$ and the result follows. □

Chapter 5.

Classification of certain maps near identity

This chapter is independent of the rest of the thesis. We will obtain here necessary and sufficient conditions for certain families of maps of the interval near the identity to be conjugate. This result will be needed in chapter 6 to finish the study of topologically versal deformations of cod 1 matrices that we started in the previous chapter. However it has also independent applications as we will see in the end of this chapter.

Consider a dynamical system which is given by the iteration of a one parameter family of maps of the half open interval $(0,a]$, $a > 0$, of the form:

$$\begin{cases} P_\theta : (0,a] \to \mathbb{R} & \theta \in (-\delta,\delta) \text{ , } \delta > 0 \\ \qquad v \to v + \theta Z_\theta(v) = v + \theta Z(\theta,v) & \end{cases}$$

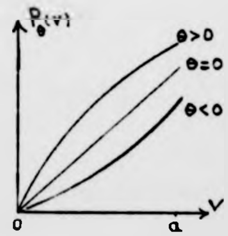


Figure 5.1

satisfying the conditions:

M(i) - the map

$$\begin{cases} Z : (-\delta,\delta) \times (0,a] \to \mathbb{R} \\ \qquad (\theta,v) \quad \to Z(\theta,v) \end{cases}$$

is C^0 in both variables.

M(ii) - $Z(\theta,v) \to 0$ as $v \to 0^+$, $\forall\ \theta \in (-\delta,\delta)$

M(iii) - $Z(\theta,v) > 0$ for $v \neq 0$ and $\forall\ \theta \in (-\delta,\delta)$.

Therefore P_θ is a family near the identity map (Figure 5.1). Note also that M(iii) is equivalent to say that

$$\left.\frac{\partial P_\theta(v)}{\partial\theta}\right|_{\theta=0} > 0 \ , \ \forall\, v \neq 0 \ .$$

For $v_0 \in (0,a]$, let $N(\theta,v_0)$ denote the number of iterations of the map P_θ in the interval $[v_0,a]$ i.e. $N(\theta,v_0)$ is the largest integer n satisfying $P_\theta^n(a) \geq v_0$ for $\theta < 0$ and $P_\theta^{-n}(a) \geq v_0$ for $\theta > 0$.

<u>Lemma 5.1</u>

$$\lim_{\theta\to 0} \theta N(\theta,v_0) = \int_{v_0}^{a} \frac{dv}{Z(0,v)} = \int_{v_0}^{a} \frac{dv}{\left.\frac{\partial P_\theta(v)}{\partial\theta}\right|_{\theta=0}} \ .$$

<u>Proof</u>

Assume first that $\theta \to 0^+$. Choose $\varepsilon > 0$. As Z is positive and continuous in $(-\delta,\delta) \times [v_0,a]$, $\exists\ \theta_0 \in (-\delta,\delta)$ such that

(1) $$\left|\frac{1}{Z(\theta,v)} - \frac{1}{Z(0,v)}\right| < \frac{\varepsilon}{3(a-v_0)} \ , \ \forall\, \theta \in [0,\theta_0] \ , \ \forall\, v \in [v_0,a] \ .$$

Furthermore as $\frac{1}{Z(0,v)}$ is integrable in $[v_0,a]$, there exists a positive integer M such that

(2) $$\left|\int_{v_0}^{a} \frac{dv}{Z(0,v)} - \sum_{i=1}^{M} \frac{h}{Z(0,x_i)}\right| < \frac{\varepsilon}{3}$$

where $h = \frac{a-v_0}{M}$ and the inequality holds for $\forall\, x_i \in [v_{i-1},v_i] =$

$= [v_0 + (i-1)h,\ v_0 + ih]$. Let $N_i(\theta)$ denote the number of iterates of P_θ in (v_{i-1}, v_i) , then we have:

(3) $$\sum_{i=1}^{M} N_i(\theta) \le N(\theta, v_0) \le \left(\sum_{i=1}^{M} N_i(\theta)\right) + M\theta \quad .$$

Put $\max\limits_{v\in[v_{i-1},v_i]} Z(\theta,v) = Z(\theta,\bar{\bar{v}}_i)$ and $\min\limits_{v\in[v_{i-1},v_i]} Z(\theta,v) = Z(\theta,\bar{v}_i)$

then we have the inequalities

$$\frac{h}{\theta Z(\theta,\bar{\bar{v}}_i)} - 1 \le N_i(\theta) \le \frac{h}{\theta Z(\theta,\bar{v}_i)} + 1 \quad .$$

Multiplying all sides by θ and summing over all intervals, this becomes:

(4) $$\left(\sum_{i=1}^{M} \frac{h}{Z(\theta,\bar{\bar{v}}_i)}\right) - M\theta \le \sum_{i=1}^{M} \theta N_i(\theta) \le \left(\sum_{i=1}^{M} \frac{h}{Z(\theta,\bar{v}_i)}\right) + M\theta \quad .$$

Combining (1) and (2) we get:

(5) $$\left(\sum_{i=1}^{M} \frac{h}{Z(\theta,\bar{\bar{v}}_i)}\right) - M\theta \le \theta N(\theta,v_0) \le \left(\sum_{i=1}^{M} \frac{h}{Z(\theta,\bar{v}_i)}\right) + 2M\theta \quad .$$

Using (1) in (5) the latter gives:

(6) $$\left(\sum_{i=1}^{M} \frac{h}{Z(0,\bar{\bar{v}}_i)}\right) - \frac{Mh\varepsilon}{3(a-v_0)} - M\theta \le \theta N(\theta,v_0) \le \left(\sum_{i=1}^{M} \frac{h}{Z(0,\bar{v}_i)}\right) + \frac{Mh\varepsilon}{3(a-v_0)} + 2M\theta \quad .$$

Substituting $Mh = (a-v_0)$ and using (2), (6) becomes

$$\int_{v_0}^{a} \frac{dv}{Z(0,v)} - \frac{2\varepsilon}{3} - M\theta < \theta N(\theta,v_0) \le \int_{v_0}^{a} \frac{dv}{Z(0,v)} - \frac{2\varepsilon}{3} + 2M\theta \quad .$$

Hence for $\theta < \text{Min}(\theta_0, \frac{\varepsilon}{6M})$ we finally get

$$\left|\theta N(\theta,v_0) - \int_{v_0}^{a} \frac{dv}{Z(0,v)}\right| < \varepsilon$$

which proves that $\lim\limits_{\theta\to 0^+} \theta N(\theta,v_0) = \int_{v_0}^{a} \frac{dv}{Z(0,v)}$. Similarly we prove $\lim\limits_{\theta\to 0^-} \theta N(\theta,v_0) = \int_{v_0}^{a} \frac{dv}{Z(0,v)}$. □

Suppose we have two dynamical systems given by the iteration of two families P_θ and $\bar{P}_\theta$ on $(0,a]$ with $P_\theta(v) = v + \theta Z(\theta,v)$ and $\bar{P}_\theta(v) = v + \theta\bar{Z}(\theta,v)$, which satisfy the conditions M(i)-M(iii) above. Then we have:

<u>Theorem 5.2</u>

P_θ and $\bar{P}_\theta$ are conjugate as one parameter families of maps (i.e. $\exists$ a family of homeomorphisms h_θ of intervals with $\bar{P}_\theta \circ h_\theta = h_\theta \circ P_\theta$ which depends continuously on θ) iff $\int_0^a \frac{dv}{Z(0,v)} = \int_0^a \frac{dv}{\bar{Z}(0,v)} \le \infty$.

<u>Proof</u>

(i) Assume first that $\int_0^a \frac{dv}{Z(0,v)} = \int_0^a \frac{dv}{Z(0,v)}$. Suppose $\theta > 0$. Clearly $P_\theta^{-1} : (0, P_\theta(a)] \to (0,a]$ is a continuous map for each $\theta \in (0,\delta)$. Let $h_\theta : (a, P_\theta(a)] \to (a, \bar{P}_\theta(a)]$ be the map taking $(a, P_\theta(a)]$ onto $(a, \bar{P}_\theta(a)]$ linearly. Extend h_θ to $h_\theta : (0, P_\theta(a)] \to (0, \bar{P}_\theta(a)]$ by the conjugacy relation $h_\theta \circ P_\theta = \bar{P}_\theta \circ h_\theta$. It is easy to see that $h_\theta(v) \to 0$ as $v \to 0^+$. The continuous dependence of h_θ on $\theta \in (0,\delta)$ is an immediate consequence of the continuity of P_θ and $\bar{P}_\theta$. Now let

$$\begin{cases} h_0 : (0,a] \to (0,a] \\ \quad\quad v \to h_0(v) \end{cases}$$

be defined by the relation $\int_{h_0(v)}^a \frac{dv}{\bar{Z}(0,v)} = \int_v^a \frac{dv}{Z(0,v)}$. Since $Z(0,v)$ and $\bar{Z}(0,v)$ are both positive for $v \in (0,a]$, h_0 is a well-defined increasing function with $h_0(a) = a$ and $h_0(v) \to 0$ as $v \to 0^+$ by our assumption. Hence h_0 is a homeomorphism of $(0,a]$ onto itself. We now prove that $h_\theta(v) \to h_0(v_0)$ as $(\theta,v) \to (0^+,v_0)$ with $v_0 \in (0,a]$. By the lemma and by the definition of h_0 we have:

$$(1) \quad \lim_{\theta \to 0^+} \theta N(\theta,v_0) \int_{v_0}^a \frac{dv}{Z(0,v)} = \int_{h_0(v)}^a \frac{dv}{\bar{Z}(0,v)}$$

$$(2) \quad \lim_{\theta \to 0^+} \theta \bar{N}(\theta,h_0(v_0)) = \int_{h_0(v)}^a \frac{dv}{Z(0,v)} \quad .$$

Proof

(i) Assume first that $\int_0^a \frac{dv}{Z(0,v)} = \int_0^a \frac{dv}{\bar{Z}(0,v)}$. Suppose $\theta > 0$. Clearly $P_\theta^{-1} : (0, P_\theta(a)] \to (0,a]$ is a continuous map for each $\theta \in (0,\delta)$. Let $h_\theta : (a, P_\theta(a)] \to (a, \bar{P}_\theta(a)]$ be the map taking $(a, P_\theta(a)]$ onto $(a, \bar{P}_\theta(a)]$ linearly. Extend h_θ to $h_\theta : (0, P_\theta(a)] \to (0, \bar{P}_\theta(a)]$ by the conjugacy relation $h_\theta \circ P_\theta = \bar{P}_\theta \circ h_\theta$. It is easy to see that $h_\theta(v) \to 0$ as $v \to 0^+$. The continuous dependence of h_θ on $\theta \in (0,\delta)$ is an immediate consequence of the continuity of P_θ and $\bar{P}_\theta$. Now let

$$\begin{cases} h_0 : (0,a] \to (0,a] \\ \qquad v \to h_0(v) \end{cases}$$

be defined by the relation $\int_{h_0(v)}^a \frac{dv}{\bar{Z}(0,v)} = \int_v^a \frac{dv}{Z(0,v)}$. Since $Z(0,v)$ and $\bar{Z}(0,v)$ are both positive for $v \in (0,a]$, h_0 is a well-defined increasing function with $h_0(a) = a$ and $h_0(v) \to 0$ as $v \to 0^+$ by our assumption. Hence h_0 is a homeomorphism of $(0,a]$ onto itself. We now prove that $h_\theta(v) \to h_0(v_0)$ as $(\theta,v) \to (0^+,v_0)$ with $v_0 \in (0,a]$. By the lemma and by the definition of h_0 we have:

$$(1) \quad \lim_{\theta\to 0^+} \theta N(\theta,v_0) \int_{v_0}^a \frac{dv}{Z(0,v)} = \int_{h_0(v)}^a \frac{dv}{\bar{Z}(0,v)}$$

$$(2) \quad \lim_{\theta\to 0^+} \theta \bar{N}(\theta,h_0(v_0)) = \int_{h_0(v)}^a \frac{dv}{Z(0,v)} \quad .$$

But our construction of h_θ implies $N(\theta,v_0) = \bar{N}(\theta,h_\theta(v_0))$ and so (1) becomes:

$$(3) \qquad \lim \theta\bar{N}(\theta,h_\theta(v_0)) = \int_{h_0(v_0)}^{a} \frac{dv}{\bar{Z}(0,v)} \quad .$$

Choose $\varepsilon > 0$ and for convenience put $f(t) = \int_t^a \frac{dv}{\bar{Z}(0,v)}$.

Then by (2):

$$\lim_{\theta\to 0^+} \frac{\theta\bar{N}(\theta,h_0(v_0)-\varepsilon)-\theta\bar{N}(\theta,h_0(v_0))}{\theta\,\bar{N}(\theta\,,h_0(v_0))} = \frac{f(h_0(v_0)-\varepsilon)-f(h_0(v_0))}{f(h_0(v_0))} = n(\varepsilon) > 0 \ .$$

So $\exists\ \theta_0 > 0$ such that

$$(4) \quad \frac{\bar{N}(\theta,h_0(v_0)-\varepsilon) - \bar{N}(\theta,h_0(v_0))}{\bar{N}(\theta,h_0(v_0))} > \frac{n(\varepsilon)}{2} \quad \text{for } \theta \in (0,\theta_0) \ .$$

On the other hand by (3) and (2):

$$(5) \quad \lim_{\theta\to 0^+} \frac{\theta\bar{N}(\theta,h_\theta(v))-\theta\bar{N}(\theta,h_0(v))}{\theta\bar{N}(\theta,h_0(v))} = \frac{f(h_0(v) - f(h_0(v_0))}{f(h_0(v_0))} \quad .$$

Since f and h_0 are both continuous $\exists\ \delta > 0$ such that $|v-v_0| < \delta \Rightarrow \left|\frac{f(h_0(v)-f(h_0(v))}{f(h_0(v_0))}\right| < \frac{n(\varepsilon)}{4}$. Now by (5), $\exists\ \theta_1 > 0$ such that

$$(6) \quad \left|\frac{\bar{N}(\theta,h_\theta(v))-\bar{N}(\theta,h_0(v))}{\bar{N}(\theta,h_0(v))}\right| < \frac{n(\varepsilon)}{2} \quad \text{for } |v-v_0| < \delta \text{ and } 0 < \theta < \theta_1 \ .$$

Hence for $|v-v_0| < \delta$ and $0 < \theta < \min(\theta_0,\theta_1)$, (4) and (6) imply:

$$\frac{\bar{N}(\theta,h_\theta(v))-\bar{N}(\theta,h_0(v_0))}{\bar{N}(\theta,h_0(v_0))} < \frac{\bar{N}(\theta,h_0(v_0)-\varepsilon)-\bar{N}(\theta,h_0(v_0))}{\bar{N}(\theta,h_0(v_0))}$$

or $\bar{N}(\theta,h_\theta(v)) < \bar{N}(\theta,h_0(v_0)-\varepsilon)$. But $\bar{N}(\theta,t)$ increases as t decreases. Therefore we finally obtain $h_\theta(v) > h_0(v_0) - \varepsilon$ for $|v-v_0| < \delta$ and $0 < \theta < \min(\theta_0,\theta_1)$. The reverse inequality is established in a similar manner. Hence $h_\theta(v) \to h_0(v_0)$ as $(\theta,v) \to (0^+,v_0)$ with $v_0 \in (0,a]$. From this, it easily follows that $h_\theta(v) \to 0$ as $(\theta,v) \to (0^+,0)$.

For $\theta < 0$, we define $h_\theta : (P_\theta(a),a] \to (\bar{P}_\theta(a),a]$ as the linear map and extend it by the conjugacy relation to $h_\theta:(0,a] \to (0,a]$. We then show by steps similar to those above that $h_\theta(v) \to h_0(v_0)$ as $(\theta,v) \to (0^-,v_0)$ with $v_0 \in (0,a]$. This will prove the sufficiency condition.

(ii) Assume $\int_0^a \frac{dv}{Z(0,v)} \neq \int_0^a \frac{dv}{\bar{Z}(0,v)}$ and suppose there exists a family h_θ of homeomorphisms of intervals which induces a conjugacy between P_θ and $\bar{P}_\theta$. Then $\exists\ v_0 \in (0,a]$ such that $\int_{v_0}^a \frac{dv}{Z(0,v)} \neq \int_{h_0(v_0)}^a \frac{dv}{\bar{Z}(0,v)}$. WLG assume $\int_{v_0}^a \frac{dv}{Z(0,v)} > \int_{h_0(v_0)}^a \frac{dv}{\bar{Z}(0,v)}$.

By the conjugacy condition we must have $N(\theta,v_0) = \bar{N}(\theta,h_\theta(v_0))$ for

$\theta \in (-\delta,\delta)$. Combining this with the lemma we obtain

$$\lim_{\theta\to 0} \theta\bar{N}(\theta,h_\theta(v_0)) = \int_{v_0}^{a} \frac{dv}{Z(0,v)}$$

$$\lim_{\theta\to 0} \theta\bar{N}(\theta,h_0(v_0)) = \int_{h_0(v_0)}^{a} \frac{dv}{\bar{Z}(0,v)} \quad .$$

Hence

$$\lim_{\theta\to 0} \frac{\bar{N}(\theta,h_\theta(v_0))-\bar{N}(\theta,h_0(v_0))}{\bar{N}(\theta,h_0(v_0))} = \frac{\int_{v_0}^{a}\frac{dv}{Z(0,v)} - \int_{h_0(v_0)}^{a}\frac{dv}{\bar{Z}(0,v)}}{\int_{h_0(v_0)}^{a}\frac{dv}{\bar{Z}(0,v)}} = A > 0 \quad .$$

So $\exists\ \theta_1 > 0$ such that

$$(7) \quad \frac{\bar{N}(\theta,h_\theta(v_0))-\bar{N}(\theta,h_0(v_0))}{\bar{N}(\theta,h_0(v_0))} > \frac{A}{2} \quad \text{for} \quad |\theta| < \theta_1 \ .$$

Since f (as defined in (i)) and h_0 are continuous, $\exists\ \varepsilon > 0$ such that $\dfrac{f(h_0(v_0)-\varepsilon) - f(h_0(v_0))}{f(h_0(v_0))} < \dfrac{A}{4}$. Therefore

$$\lim_{\theta\to 0} \frac{\bar{N}(\theta,h_0(v_0)-\varepsilon)-\bar{N}(\theta,h_0(v_0))}{\bar{N}(\theta,h_0(v_0))} = \frac{f(h_0(v_0)-\varepsilon)-f(h_0(v_0))}{f(h_0(v_0))} < \frac{A}{4}$$

and $\exists\ \theta_2 > 0$ such that

$$(8) \quad \frac{\bar{N}(\theta,h_0(v_0)-\varepsilon)-\bar{N}(\theta,h_0(v_0))}{\bar{N}(\theta,h_0(v_0))} < \frac{A}{2} \quad \text{for} \quad |\theta| < \theta_2 \ .$$

It then follows by (7) and (8) that for $|\theta| < \min(\theta_1,\theta_2)$ we have:

$$\frac{\bar{N}(\theta,h_\theta(v_0))-\bar{N}(\theta,h_0(v_0))}{\bar{N}(\theta,h_0(v_0))} > \frac{A}{2} > \frac{\bar{N}(\theta,h_0(v_0)-\varepsilon)-\bar{N}(\theta,h_0(v_0))}{\bar{N}(\theta,h_0(v_0))} .$$

So $\bar{N}(\theta,h_\theta(v_0)) > \bar{N}(\theta,h_0(v_0)-\varepsilon)$ for $|\theta| < \min(\theta_1,\theta_2)$ from which we obtain $h_\theta(v_0) < h_0(v_0)-\varepsilon$ for $|\theta| < \min(\theta_1,\theta_2)$. But this contradicts the continuity of $h_\theta(v_0)$ as $\theta \to 0$. Therefore the necessary condition is also established. □

Note that $\int_0^a \frac{dv}{Z(0,v)}$ is the time taken to flow from $v = a$ to $v = 0$ for the vector field $\dot{v} = Z(0,v)$. The theorem states that this time is invariant under conjugacy.

<u>Corollary 5.3</u>

Allowing a C^1-reparametrization $\theta \to \eta(\theta)$ with $\eta'(0) \neq 0$, P_θ and $\bar{P}_\theta$ are conjugate as one parameter families iff $\int_0^a \frac{dv}{Z(0,v)}$ and $\int_0^a \frac{dv}{\bar{Z}(0,v)}$ are both finite or both infinite.

<u>Proof</u>

(i) If both integrals are infinite then P_θ and $\bar{P}_\theta$ are conjugate by the theorem. Suppose $\int_0^a \frac{dv}{Z(0,v)} = \alpha$ and $\int_0^a \frac{dv}{\bar{Z}(0,v)} = \bar{\alpha}$ with α and $\bar{\alpha}$ both finite. Let $\eta:\theta \to \eta(\theta)$ be given by $\eta(\theta) = \frac{\bar{\alpha}}{\alpha}\theta$ then P_θ is conjugate to $\bar{\bar{P}}_\theta = \bar{P}_{\eta(\theta)}$.

(ii) Suppose P_θ is conjugate to $\bar{P}_{n(\theta)}$, with $n'(0) \neq 0$.

Writing $\bar{P}_{n(\theta)} : v \to v + n(\theta)\,\bar{Z}(n(\theta),v) = v + \theta\,\frac{n(\theta)}{\theta}\,\bar{Z}(n(\theta),v)$,

the theorem gives $\int_0^a \frac{dv}{Z(0,v)} = \int_0^a \frac{n'(0)}{\bar{Z}(0,v)}\,dv$. Hence the two original integrals are both finite or both infinite. □

Remark 5.4

Although h_θ as constructed in the theorem is only piecewise differentiable for $\theta \neq 0$, h_0 is differentiable in the whole interval $(0,a]$. In fact by differentiating the integral defining h_0 we get $h_0'(v) = \frac{\bar{Z}(0,h_0(v))}{Z(0,v)}$ for $v \in (0,a]$. If P_θ and $\bar{P}_\theta$ and their inverses are C^2, we can construct the conjugacy map h_θ such that it is C^1 in $(0,a]$ for $\forall\,\theta \in (-\delta,\delta)$ and $\lim_{\theta\to 0} h_\theta'(v) = h_0(v)$ for $v \in (0,a]$. We will not give details and will only highlight the method. First note that h_θ $(\theta \neq 0)$ can be chosen to be C^2. This can be done, for example, by taking $h_\theta : (a, P_\theta(a)] \to (a,\bar{P}_\theta(a)]$, for $\theta > 0$, to be a polynomial (rather than the linear map) with values of h_θ, h_θ' and h_θ'' at $v = a$ matching with the corresponding values at $v = P_\theta(a)$ through the relation $\bar{P}_\theta \circ h_\theta = h_\theta \circ P_\theta$. We can then extend h_θ to the interval $(0,P_\theta(a)]$ by the conjugacy relation as before and do a similar construction for $\theta < 0$. h_0 is defined as before. Next it can be proved that given $v_0 \in (0,a]$, $\exists\, K = K(v_0)$ such that $|h_\theta''(v_1)| < K$, for $\forall\,\theta \in (-\delta,\delta)$ and $v_1 \in [v_0,a]$.

Writing the conjugacy relation in terms of Z and $\bar{Z}$, we get $v + \theta\bar{Z}(\theta,h_\theta(v)) = h_\theta(v + \theta Z(\theta,v))$ and a Taylor expansion gives

$$h_\theta'(v) = \frac{\bar{Z}(\theta,h\ (v))}{Z(\theta,v)} - \frac{\theta}{2} Z^2(\theta,v)h_\theta''(v_1)\ , \quad v-\theta Z(\theta,v) < v_1 < v \quad .$$

Since $h_\theta''(v_1)$ is uniformly bounded we deduce that

$\lim_{\theta\to 0}\left|h_\theta'(v) - \frac{\bar{Z}(\theta,h_\theta(v))}{Z(\theta,v)}\right| = 0$ from which it follows that

$\lim_{\theta\to 0} |h_\theta'(v) - h_0'(v)| = 0$. □

Returning to theorem 5.2 and its corollary, note that we have proved that there are up to reparametrization two conjugacy classes of families of maps P_θ satisfying conditions M(i)-M(iii). Taking the simple example $P_\theta(v) = v + \theta Z(\theta,v) = v + \theta v^n$, $n > 0$, we see that families with $n \geq 1$ fall into one conjugacy class and those with $0 < n < 1$ fall into the other class.

As a more interesting application, consider the family of planar vector fields which in polar coordinates is given by:

$$\begin{cases} \dot{r} = \theta r^n \\ \dot{\psi} = 1 \end{cases} \qquad n > 0$$

The phase portraits for $\theta < 0$, $\theta = 0$ and $\theta > 0$ are as in Figure 3.1. Now we ask the question: When are two families of this type equivalent?

Proposition 5.5

There are two equivalent classes of the families $\begin{cases} \dot{r} = \theta r^n \\ \dot{\psi} = 1 \end{cases}$ $n > 0$, one corresponding to $n \geq 1$ the other to $0 < n < 1$.

Proof

Take the semi-infinite line $\psi = 0$ as the Poincare section. Integrating $\frac{dr}{d\psi} = \theta r^n$ from $\psi = 0$ to 2π we can find explicitly the return map: $P_\theta : [0,\infty) \to [0,\infty)$

$$r \to \begin{cases} r(1-2(n-1)\pi\theta r)^{-\frac{1}{n-1}} & n \neq 1 \\ r \exp 2\pi\theta & n = 1 \end{cases}$$

Hence we obtain $Z(\theta,r) = r[\dfrac{(1-2(n-1)\pi\theta r^{n-1})^{-\frac{1}{n-1}} - 1}{\theta}]$ for $n \neq 1$ and $Z(\theta,r) = r(\frac{\exp 2\pi\theta - 1}{\theta})$ for $n = 1$. Therefore $Z(0,r) = 2\pi r^n$.

Now consider the families $\begin{cases} \dot{r} = \theta r^n \\ \dot{\psi} = 1 \end{cases}$ and $\begin{cases} \dot{r} = \theta r^{\bar{n}} \\ \dot{\psi} = 1 \end{cases}$ with $n, \bar{n} \geq 1$ (respectively $n, \bar{n} \in (0,1)$). Since $\int_0^\infty \frac{1}{r^{\bar{n}}} dr = \infty$ for $\forall\, \bar{n} > 0$, $\exists\, a \in (\theta,\infty)$ such that $\int_1^\infty \frac{1}{r^n} dr = \int_a^\infty \frac{1}{r^{\bar{n}}} dr$ (respectively $\int_0^1 \frac{1}{r^n} dr = \int_0^a \frac{1}{r^{\bar{n}}} dr$). Construct $h_\theta : [0,\infty) \to [0,\infty)$ as follows. For $\theta > 0$, extend the linear map $h_\theta : (1,P_\theta(1)] \to (a,\bar{P}_\theta(a)]$ by the conjugacy condition $\bar{P}_\theta \circ h_\theta = h_\theta \circ P_\theta$. For $\theta < 0$, do the same with the linear map $h_\theta : (P_\theta(1),1] \to (\bar{P}_\theta(a),a]$.

Define $\begin{cases} h_0 : [0,\infty) \to [0,\infty) \\ \qquad r \to h_0(r) \end{cases}$ by $\displaystyle\int_a^{h_0(r)} \frac{1}{r^{\bar{n}}}\,dr = \int_1^r \frac{1}{r^n}\,dr$.

Then h_θ , $\theta \in \mathbb{R}$, will be a family of homeomorphisms of $[0,\infty)$ onto itself which depends continuously on θ since, by our choice of a , $\displaystyle\int_1^\infty \frac{1}{r^n}\,dr = \int_a^\infty \frac{1}{r^{\bar{n}}}\,dr < \infty$ and $\displaystyle\int_0^1 \frac{1}{r^n} = \int_0^a \frac{1}{r^{\bar{n}}}\,dr = \infty$ (respectively $\displaystyle\int_1^\infty \frac{1}{r^n}\,dr = \int_a^\infty \frac{1}{r^{\bar{n}}}\,dr = \infty$ and $\displaystyle\int_0^1 \frac{1}{r^n}\,dr = \int_0^a \frac{1}{r^{\bar{n}}}\,dr < \infty$) . We now extend h_θ to $H_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by sending the orbit segment from $(r,0)$ to $(P_\theta(r),0)$ of the vector field $\begin{cases} \dot{r} = \theta r^n \\ \dot{\psi} = 1 \end{cases}$ onto the orbit segment from $(h_\theta(r),0)$ to $(\bar{P}_\theta \circ h_\theta(r),0)$ of the vector field $\begin{cases} \dot{r} = \theta r^{\bar{n}} \\ \dot{\psi} = 1 \end{cases}$ arcwise linearly. Then H_θ induces an equivalence between the two vector field families.

Conversely, if $H_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ induces an equivalence between the two families, then h_θ , the restriction of H_θ to $[0,\infty)$, induces a conjugacy between the family of return maps P_θ and $\bar{P}_\theta$. Putting $a = h_0(1)$, we must have $\displaystyle\int_0^1 \frac{1}{r^n}\,dr = \int_0^a \frac{1}{r^{\bar{n}}}\,dr \le \infty$ and $\displaystyle\int_1^\infty \frac{1}{r^n}\,dr = \int_a^\infty \frac{1}{r^{\bar{n}}}\,dr \le \infty$. Therefore either $n,\bar{n} \ge 1$ or $n,\bar{n} \in (0,1)$. □

The above proposition can be generalized as follows. Consider smooth families of vector fields of the form $\begin{cases} \dot{r} = \theta r^n + f(r,\psi,\theta) \\ \dot{\psi} = g(\theta) + h(r,\psi,\theta) \end{cases}$

which satisfy the conditions: (i) $\lim\limits_{r\to 0} \dfrac{f(r,\psi,\theta)}{r^{\bar{n}}} = \lim\limits_{r\to 0} h(r,\psi,\theta) = 0$.

(ii) $g(0) \neq 0$, and (iii) At $\theta = 0$, all orbits in a neighbourhood of origin are closed. Then two such families are locally equivalent near the origin iff $n,\bar{n} \geq 1$ or $n,\bar{n} < 1$. The method for proving this is as in theorem 5.6 below which is of more practical interest.

Theorem 5.6

Consider the smooth families of planar vector fields with a fixed point at origin which satisfy the two conditions: (i) The eigenvalues the origin are distinct complex conjugate pairs which cross the imaginary axis transversally at the bifurcation point. (ii) At the bifurcation point all orbits in a neighbourhood of the origin are closed. Then any two such families are locally equivalent near the origin.

Proof

By a reparametrization we can assume that the eigenvalues at the origin are $\theta \pm ig(\theta)$ where $g(0) \neq 0$. Then the family can be written in the form (see [14]):

$$\begin{cases} \dot{r} = \theta r^n + f(r,\psi,\theta) \\ \dot{\psi} = g(\theta) + h(r,\psi,\theta) \end{cases}$$

with $\lim\limits_{r\to 0} \dfrac{f(r,\psi,\theta)}{r^{\bar{n}}} = \lim\limits_{r\to 0} h(r,\psi,\theta) = 0$. The return map P_θ is well-defined on $\theta = 0$ in a neighbourhood of the origin. Let $\tau_\theta(r)$ denote the time of flow from $(r,0)$ to $(P_\theta(r),0)$ for the above vector field. Then $\tau_\theta(r) \sim g(\theta)$ as $r \to 0$ [14]. To obtain an estimate for $P_\theta(r)$ as $r \to 0$ we integrate $\frac{\dot{r}}{r}$ along the orbit from $(r,0)$ to $(P_\theta(r),0)$:

$$\log \frac{P_\theta(r)}{r} = \int_0^{\tau_\theta(r)} \frac{\dot{r}}{r}\, dt = \theta\tau_\theta(r) + \int_0^{\tau_\theta(r)} \frac{f(r,\psi,\theta)}{r}\, dt \quad .$$

Hence we can write $\log \frac{P_\theta(r)}{r} = \theta\tau_\theta(r) + R(r,\theta)$ where R is a smooth function of (r,θ) and $R(r,\theta) \to 0$ as $r \to 0$. But at $\theta = 0$, all orbits are closed in a neighbourhood of origin. Therefore $R(r,\theta) = \theta R_1(r,\theta)$ where R_1 is smooth and $R_1(r,\theta) \to 0$ as $r \to 0$. This implies $P_\theta(r) = r \exp\theta[\tau_\theta(r) + R_1(r,\theta)]$ from which we get $Z(0,r) = \left.\frac{\partial P_\theta(r)}{\partial\theta}\right|_{\theta=0} = r(\tau_0(r) + R_1(r,0))$ where $\tau_0(r) \to g(0)$ as $r \to 0$. Hence conditions M(i)-M(iii) holds and $\int_0^r \frac{dr}{Z(0,r)}$ diverges. The result then follows as in proposition 5.5. □

Finally, we note that the generic case for families of vector fields with a pair of distinct complex conjugate eigenvalues crossing the imaginary axis transversally is that of Hopf bifurcation where a stable limit cycle is born. Any two such generic families are locally equivalent which can be proved by reducing the family to normal form. [5] [12]

Chapter 6.

Versal deformations of cod 1 matrices (Part two)

In this chapter we will finish off the study of topologically versal deformations of cod 1 matrices by dealing with the remaining three cases which induce cycles of saddles in Δ . Considerable difficulty arises in these cases as always when a cycle of saddles exists in a dynamical system. We will need the results of chapter 5 for our proofs in this chapter.

6.1 Deformations of I_1 (The Hypercycle)

The flow induced by a cod 1 matrix $A \in I_1$ is called a hypercycle which together with its analogue in higher dimensions has been extensively treated in the literature. We have already studied the bifurcation induced by a transversal deformation of such a matrix in proposition 3.1. Here we will prove that a transversal deformation $(0,f)$ of A is topologically versal. Let (U,f) be a representative unfolding which as in proposition 3.1 we assume is in central form

$$f(\theta) = \begin{pmatrix} 0 & \theta+a_1(\theta) & \theta-a_1(\theta) \\ \theta-a_2(\theta) & 0 & \theta+a_2(\theta) \\ \theta+a_3(\theta) & \theta-a_3(\theta) & 0 \end{pmatrix} , \; 0 \le |\theta| < a_i(\theta) \; , \; \theta \in U \; .$$

We also assume that $U = (-\delta,\delta)$ with $\delta > 0$. Consider the open interval $\overset{\circ}{\overline{EJ}}$ where E is the barycentre of Δ and J is the midpoint of $\overline{X_1X_2}$. A lemma in [25] shows that orbits in $\overset{\circ}{\Delta}\backslash E$ cross all rays through E transversally. Hence the poincare return map P_θ of the flow $\Delta_{f(\theta)}$ is well defined on $\overset{\circ}{\overline{EJ}}$ for all $\theta \in (-\delta,\delta)$. Suppose G is, say, the midpoint of $\overline{EJ}$ (Figure 6.1). Taking $y = 3x_3$ as the coordinate on $\overline{EJ}$ we can write P_θ as:

$$\begin{cases} P_\theta : (0,1) \to (0,1) \\ \qquad\quad y \mapsto y + R(\theta,y) \end{cases}$$

where R is a smooth function of $(\theta,y) \in (-\delta,\delta) \times (0,1)$. Since at $\theta = 0$ all orbits in $\overset{\circ}{\Delta}$ are closed we have $R(0,y) = 0$ for $\forall\, y \in (0,1)$. Therefore $R(\theta,y) = \theta Z(\theta,y)$ where Z is again a smooth function of (θ,y) . Hence $P_\theta(y) = y + \theta Z(\theta,y)$. We now ask whether the two families $P_\theta\big|_{(0,\frac{1}{2}]}$ and $P_\theta\big|_{[\frac{1}{2},1)}$ each satisfy the conditions M(i)-M(iii) of chapter 5. In fact we have just seen that M(i) holds and inspection shows that M(ii) holds as well. However we cannot prove that M(iii) holds in general; although this can be shown when a_i's do not depend on θ (see remark 6.2).

Figure 6.1

In order to be able to apply the results of chapter 5, we make a coordinate transformation as follows. Recall from proposition 3.1 that

$$\begin{cases} V_\theta : \overset{\circ}{\Delta} \to (0,1] \\ x \to \dfrac{\prod_i x_i^{b_i(\theta)}}{\Sigma\, b_i(\theta) x_i} \end{cases} .$$

The restriction of V_θ to the open interval $\overline{\overset{\circ}{EJ}}$, which we also denote by V_θ, can be written in terms of $y = 3x_3$ as

$$\begin{cases} V_\theta : (0,1) \to (0,1) \\ y \to \dfrac{2^{b_3(\theta)}\, y^{b_3(\theta)}\, (3-y)^{1-b_3(\theta)}}{(3b_3(\theta)-1)y + 3(1-b_3(\theta))} \end{cases} .$$

This gives a family of diffeomorphisms depending continuously on θ. In fact by differentiation we get

$$\frac{dV_\theta(y)}{dy} = k\, \frac{y^{b_3(\theta)-1}\,(3-y)^{-b_3(\theta)}\; b_3(\theta)(1-b_3(\theta))(1-y)}{[(3b_3(\theta)-1)y + 3(1-b_3(\theta))]^2} > 0$$

(k is a positive constant). With respect to the new coordinate system the return map takes the form

$$\tilde{P}_\theta : \begin{cases} (0,1) \to (0,1) \\ v \to \tilde{P}_\theta(v) = V_\theta \circ P_\theta \circ V_\theta^{-1}(v) \end{cases} .$$

As the time derivative of V_θ along orbits in $\mathring{\Delta}$ is given by

$$\dot{V}_\theta(x) = \theta W_\theta(x) \quad \text{where} \quad W_\theta(x) = \frac{\prod_i x_i^{b_i(\theta)} \sum_{i<j} b_i(\theta)b_j(\theta)(x_i-x_j)^2}{(\Sigma\, b_i(\theta)x_i)^2} > 0$$

$(x \in \mathring{\Delta})$, we can write $\tilde{P}_\theta$ in the form

$$\tilde{P}_\theta(v) = v + \int_0^{\tau_\theta(v)} \dot{V}_\theta(x)dt = v + \theta \int_0^{\tau_\theta(v)} W_\theta(x)dt = v + \theta\tilde{Z}(\theta,v)$$

where the integrals are taken along the orbit of $\dot{\Delta}_{f(\theta)}$ from $V_\theta^{-1}(v)$ on $\overset{o}{\overline{EJ}}$ to the first return at $P_\theta \circ V_\theta^{-1}(v)$, and $\tau_\theta(v)$ is the corresponding time of flow. Clearly conditions M(i) and M(ii) are satisfied by $\tilde{P}_\theta\big|_{(0,\frac{1}{2}]}$ and $\tilde{P}_\theta\big|_{[\frac{1}{2},1)}$. Since $W_\theta(x) > 0$ for $\forall\, x \in \Delta$, $\forall\, \theta \in (-\delta,\delta)$, we find that M(iii) holds as well. Now we deduce

<u>Proposition 6.1</u>

$$\text{(i)} \quad \int_0^{\frac{1}{2}} \frac{dv}{\tilde{Z}(0,v)} = \infty \qquad\qquad \text{(ii)} \quad \int_{\frac{1}{2}}^1 \frac{dv}{\tilde{Z}(0,v)} = \infty \;.$$

<u>Proof</u>

(i) We claim that $\tilde{Z}(0,v) = 0\,(-v \log v)$ as $v \to 0^+$. First note that $\exists\; K > 0$ such that $W_0(x) \le K\, V_0(x)$, $\forall\, x \in \mathring{\Delta}$. In fact

$$\frac{W_0(x)}{V_0(x)} = \frac{\sum_{i<j} b_i(0)b_j(0)\,(x_i-x_j)^2}{\sum_i b_i(0)x_i}$$ where the numerator is bounded and

the denominator is positive in the compact region Δ and hence bounded from below by a positive number. Next we need an estimate for $\tau_0(v)$ as $v \to 0^+$. For this we need to calculate the time for the flow to pass by the three saddles of $\Delta_{f(0)}$. As these saddles are hyperbolic, it is possible to C^1-linearize the vector field in a neighbourhood of each. [6] The linearized vector field in a neighbourhood of X_1 for example will be

$$\begin{cases} \dot{x}_2 = -a_2(0)x_2 \\ \dot{x}_3 = a_3(0)x_3 \end{cases}$$

In this linear system the time taken to flow from the point (x_2,x_3), with $x_3 \ll 1$ and x_2 of order 1, to the point (x_2',x_3'), with $x_2' \ll 1$ and x_3' of order 1, is easily calculated to be $-\dfrac{1}{a_3(0)} \log x_3 + O(1)$ as $x_3 \to 0^+$. The time of passing by the saddle X_1 for the nonlinear vector field is therefore of this order. Since $V_0 \sim c\, y^{b_3(0)}$ as $y \to 0^+$, where c is a positive constant, this time reduces to $-\dfrac{1}{b(0)} \log v + O(1)$ as $v \to 0^+$ (remember that $a_i b_i = b$). Since orbits of $\Delta_{f(0)}$ are the closed level curves $V_0 =$ constant we get the same expression for the time of passing by X_2 and X_3. Hence $\tau_0(v) = -\dfrac{1}{b(0)} \log v + O(1)$ as $v \to 0^+$. This implies

$$\tilde{Z}(0,v) = \int_0^{\tau_0(v)} W_0(x)dt \le K\int_0^{\tau_0(v)} V_0(x)dt = K\int_0^{\tau_0(v)} vdt = Kv\int_0^{\tau_0(v)} dt$$

$$= K\, v\, \tau_0(v) = O(-v \log v) \qquad \text{as} \quad v \to 0^+$$

which proves our claim. It now follows that $\int_0^{\frac{1}{2}} \frac{dv}{\tilde{Z}(0,v)} = \infty$.

(ii) We claim that $\tilde{Z}(0,v) = O(1-v)$ as $v \to 1^-$. The main part of the proof is to show that $\exists\ L > 0$ such that $W_0(x) \le L(1-V_0(x))$ for all x in a neighbourhood of E . For this we need the first terms in the Taylor series expansion of $1-V_0(x)$ and $W_0(x)$ around E . Putting $x_i = \frac{1}{3} + \bar{x}_i$ and remembering that $\sum_i b_i = 1$, we obtain:

$$1-V_0(x) = \frac{-\prod_i x_i^{b_i} + \Sigma b_i x_i}{\sum_i b_i x_i} = \frac{-\prod_i (1+\bar{x}_i)^{b_i} + 1 + \sum_i b_i \bar{x}_i}{1 + \Sigma b_i \bar{x}_i} = \frac{1}{2} \sum_i b_i(1-b_i \bar{x}_i^2) - \sum_{i<j} b_i b_j \bar{x}_i \bar{x}_j + \text{h.O.t.} =$$

$$\bar{x}_1^2[(b_1+b_3)-(b_1-b_3)^2]+\bar{x}_2[(b_2+b_3)-(b_2-b_3)^2]+2\bar{x}_1\bar{x}_2[2(b_1+b_2)-2(b_1+b_2)^2-b_1b_2]+\text{h.O.t.}$$

where $b_i = b_i(0)$ and h.O.t. indicates higher order terms in $\bar{x}_i$. The coefficients of $\bar{x}_1^2$ and $\bar{x}_2^2$ are both positive, since for example $(b_1+b_3) - (b_1-b_3)^2 = 4b_1(1-b_1-b_2) + b_2(1-b_2) > 0$. As for the sign of the discriminant of the quadratic we will show in appendix 3(a) that $\Delta = -9\, b_1 b_2 b_3 < 0$. Therefore the quadratic is positive definite. Next we expand $W_0(x)$ in powers of $\bar{x}_1$ and $\bar{x}_2$ and we get:

$$W_0(\bar{x}) = \frac{1}{3}[b_1b_2(\bar{x}_1-\bar{x}_2)^2+b_1b_3(2\bar{x}_1+\bar{x}_2)^2+b_2b_3(\bar{x}_1+2\bar{x}_2)^2] + \text{h.O.t.}$$

where the quadratic in brackets is clearly positive definite. Hence in the expression $\dfrac{W_0(x)}{1-V_0(x)}$ the dominant terms in both the numerator and the denominator are positive definite quadratic and hence this expression is bounded above by a positive number in a neighbourhood of E . This establishes the existence of L . On the other hand $\tau_0(v) \to \sqrt{\rho(o)}$ as $v \to 1^-$ (see proposition 1.4 and theorem 5.6). Hence:

$$\tilde{Z}(0,v) = \int_0^{\tau_0(v)} W_0(x)dt \le L \int_0^{\tau_0(v)} 1-V_0(x)dt = L(1-v)\tau_0(v) = 0\ (1-v)$$

as $v \to 1^-$, and the result follows again. □

<u>Remark 6.3</u>

Suppose $a_i(\theta) = a_i$, $\forall\ \theta\ \ (-\delta,\delta)$, then $\tilde{P}_\theta = V_0 \circ P_\theta \circ V_0^{-1}$ and hence $\left.\dfrac{\partial\tilde{P}_\theta(v)}{\partial\theta}\right|_{\theta=0} = V_0'(V_0^{-1}(v))\ \dfrac{\partial P_\theta}{\partial\theta}\ (V_0^{-1}(v)\Big|_{\theta=0}$. Since $V_0'(y) > 0$ for $\forall\ y \in (0,1)$, it follows that $\left.\frac{\partial P_\theta}{\partial\theta}(y)\right|_{\theta=0} > 0$ and conditions M(i)-M(iii) are satisfied for $P_\theta\Big|_{(0,\frac{1}{2}]}$ and $P_\theta\Big|_{[\frac{1}{2},1)}$. In this case the divergence of $\displaystyle\int_{\frac{1}{2}}^{1} \frac{dy}{Z(0,y)}$ follows immediately from theorem 5.6. To prove the divergence of $\displaystyle\int_0^{\frac{1}{2}} \frac{dy}{Z(0,y)}$ without resort to Lyapunov functions, we need to find an approximation for $P_\theta(y)$ when y is small. If there are no resonant conditions up to order 4 in the eigenvalues $\pm\ a_i$ $(i=1,2,3)$ of the saddles of $\Delta_{f(0)}$ [i.e. if the relation $a_i = k_1a_i - k_2a_j$ does not hold for any integers $k_1 \geq 0$, $k_2 \geq 0$ with $k_1 + k_2 \leq 4$, which is equivalent to say that $a_i \neq a_j$ and $a_i \neq 2a_j$ $(i,j = 1,2,3)$], then by a theorem in [6] a C^2-linearization of neighbourhoods of the saddles of the family $\Delta_{f(\theta)}$, $\theta \in (-\delta,\delta)$, is possible. We can then show, after some work, that for sufficiently small y and θ, P_θ can be put in the form

$$P_\theta(y) = (1 + \theta k(\theta,y))y^{\prod_i \frac{a_i-\theta}{a_i+\theta}}$$

where k is a continuous function of (θ,y). From this we get

$$Z(0,y) = \lim_{\theta\to 0} \frac{P_\theta(y)-y}{\theta} = -\frac{2}{b}\, y \log y + y\, k(0,y) = 0\,(-y \log y) \quad \text{as}$$

$y \to 0^+$, in agreement with the limiting behaviour of $\tilde{Z}(0,v)$ in part one of the above proposition. However when the resonant conditions do hold, a C^2-linearization may not be possible and a C^1-linearization (which is always possible for a family of planar vector fields in a neighbourhood of a fixed point) does not enable us to determine the above limit. □

Theorem 6.3

Let $(0,f)$ and $(0,\bar{f})$ be transversal deformations of cod 1 matrices $f(0)$, $\bar{f}(0) \in I_1$ respectively. Then $(0,f)$ and $(0,\bar{f})$ are equivalent.

Proof

Let (U^*, f) and $(U^*, \bar f)$ be representative unfoldings with $U^* = (-\delta, \delta)$ as in section 4.2. Denote by P_θ , $\bar P_\theta$ $(\theta \in U^*)$ the return maps induced by $f(\theta)$ and $\bar f(\theta)$ on $\mathring{\overline{EH}}$. Put $\tilde P_\theta = V_\theta \circ P_\theta \circ V_\theta^{-1}$ and $\tilde{\bar P}_\theta = \bar V_\theta \circ \bar P_\theta \circ \bar V_\theta^{-1}$ with $\tilde P_\theta(v) = v + \theta \tilde Z(\theta, v)$ and $\tilde{\bar P}_\theta = v + \theta \tilde{\bar Z}(\theta, v)$. For $\theta \in U^* \backslash 0$, let $h_\theta : [\tfrac12, \tilde P_\theta(\tfrac12)) \to [\tfrac12, \tilde{\bar P}(\tfrac12))$ be the linear map and extend it by the conjugacy relation $h_\theta \circ \tilde P_\theta = \tilde{\bar P}_\theta \circ h_\theta$ to a map of $[0,1]$ onto $[0,1]$ with $h_\theta(0) = 0$ and $h_\theta(1) = 1$. Also define $h_0 : [0,1] \to [0,1]$

by $\displaystyle\int_{\frac12}^{h_0(v)} \frac{dv}{\tilde{\bar Z}(0,v)} = \int_{\frac12} \frac{dv}{\tilde Z(0,v)}$ for $v \in [0,1]$. Then theorem 5.2, on the basis of proposition 6.1, implies that $h_\theta : [0,1] \to [0,1]$ is a family of homeomorphisms depending continuously on $\theta \in U^*$ with $\tilde{\bar P}_\theta \circ h_\theta = h_\theta \circ \tilde P_\theta$. We can now extend h_θ to a map $H_\theta : \Delta \to \Delta$ by the technique of proposition 5.5 in $\mathring\Delta$ and by putting $H_{\theta|} = \text{Identity}$; the continuity of H_θ at $\partial\Delta$ is then ensured as in Theorem 4.5 by remark 3.7. Alternatively we can construct H_θ as a radial homeomorphism as in [2 5] i.e. by sending every ray through E onto itself such that orbits are mapped onto orbits. Either way, H_θ will induce an equivalence between $f(\theta)$ and $\bar f(\theta)$ which depends continuously on θ .

□

Note that by remark 5.4, we can construct h_θ such that it is differentiable in $(0,1)$. By extending such h_θ to a radial homeomorphism we will obtain a family H_θ which is also differentiable in $\mathring\Delta \backslash E$.

Corollary 6.4

A transversal deformation of $A \in I_1$ is topologically versal and A has "codimension" one. □

6.2 Deformation of I_2

In this section we will prove a similar result for I_2. Let $(0,f)$ be a transversal deformation of $f(0) \in I_2$ and (U,f) with $U = (-\delta,\delta)$ a representative unfolding. Notations are as in proposition 3.2 and section 6.1.

Lemma 6.5

Orbits of $\Delta_{f(\theta)}(\theta \in U)$ in $\overset{\circ}{\Delta}\setminus E$ intersect the line $x_i = x_{i+1}$ transversally $(i = 1,2,3)$.

Proof

We will show that the cross product $(x-\frac{1}{3}u) \wedge \dot{x}$, where $\frac{u}{3} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is the position vector of E, never vanishes on $x_i = x_{i+1}$. Clearly $(x - \frac{1}{3}u) \wedge \dot{x} = t(x)u$ where $t(x)$ is a scalar function and is given by

$$t(x) = [(x_i-\tfrac{1}{3})\dot{x}_{i+1}-(x_{i+1}-\tfrac{1}{3})\dot{x}_i] = x_i(x_i-\tfrac{1}{3})\,[(\theta+a_{i+1}(\theta))x_{i+2}$$

$$+ (\theta-a_{i+1}(\theta))x_i-(\theta-a_i(\theta))x_{i+2}-(\theta+a_i(\theta))x_{i+1}] = x_i(x_i-\tfrac{1}{3})[$$

$$(a_{i+1}+a_i)(1-2x_i)-(a_i+a_{i+1})x_i] = -3\,x_i(x_i-\tfrac{1}{3})^2(a_i+a_{i+1}) < 0$$

for $x \in \overset{\circ}{\Delta}\setminus E$ with $x_i = x_{i+1}$, because $a_{i+2}(\theta) + a_{i+1}(\theta) > 0$, $i = 1,2,3$, $\forall\, \theta \in (-\delta,\delta)$ (see proposition 3.2). □

Consider the section $\overset{\circ}{\overline{EF}}$, where F is the midpoint of $\overline{X_2X_3}$ (Figure 6.2). Take coordinate $y = 3x_1$ on $\overline{EF}$.

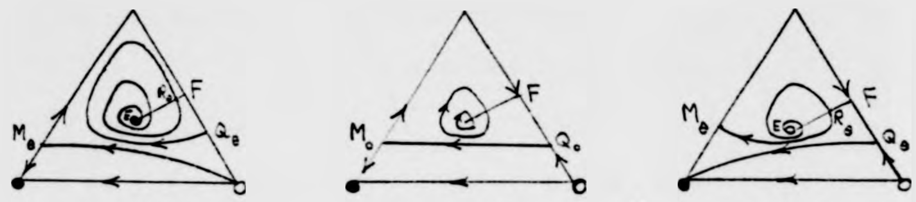


Figure 6.2

Let R_θ be the first intersection of the outset of Q_θ with $\overline{EF}$ for $\theta > 0$ and the first intersection of the inset of M_θ with $\overline{EF}$ for $\theta < 0$. Denote the y coordinate of R_θ by y_θ. Then the return map on $\overset{\circ}{\overline{EF}}$ is given by

$$\begin{cases} P_\theta : (0,1) \to (0,y_\theta) \\ \qquad\quad y \to y + \theta Z(\theta,y) \end{cases} \qquad \theta > 0$$

and

$$\begin{cases} P_\theta : (0,y_\theta) \to (0,1) \\ \qquad\quad y \to y + \theta Z(\theta,y) \end{cases} \qquad \theta < 0$$

and P_0 = Identity. Recall from proposition 3.2 that

$V_\theta(x) = (\prod_i x_i^{-b_i(\theta)}) \sum_i b_i(\theta)x_i$, $x \in \overset{\circ}{\Delta}$, with $b_3(\theta) > 0$ and $b_1(\theta)$, $b_2(\theta) < 0$. The restriction of V_θ to $\overset{\circ}{\overline{EF}}$ is

$$\begin{cases} V_\theta : (0,1) \to (0,1) \\ \quad y \to \dfrac{(2y)^{-b_1(\theta)}[(3b_1(\theta)-1)y+3(1-b_1(\theta))]}{(3-y)^{1-b_1(\theta)}} \end{cases}$$

which represents a family of diffeomorphisms depending continuously on $\theta \in (-\delta,\delta)$. As in the previous section we put $\tilde{P}_\theta = V_\theta \circ P_\theta \circ V_\theta^{-1}$. Let $\ell_\theta = V_\theta(y_\theta)$, then $\tilde{P}_\theta$ can be written as:

$$\begin{cases} \tilde{P}_\theta : (0,1) \to (\ell_\theta,1) & \theta > 0 \\ \quad (\ell_\theta,1) \to (0,1) & \theta < 0 \\ \quad v \to v + \theta\tilde{Z}(\theta,v) \end{cases}$$

where $\theta\tilde{Z}(\theta,\ ) = \int_0^{\tau_\theta(v)} V_\theta(x)dt = \theta \int_0^{\tau_\theta(v)} W_\theta(x)dt$ with

$W_\theta(x) = -(\prod_i x_i^{-b_i(\theta)}) \sum_{i<j} b_i(\theta)b_j(\theta)\,(x_i-x_j)^2 > 0$ for $x \in \mathring{\Delta}\backslash E$.

Clearly $\tilde{Z}$ is a continuous function of $(\theta,v) \in (-\delta,\delta) \times (0,1)$ and $Z(0,v) > 0 \ \forall\, v \in (0,1)$. We now prove:

<u>Proposition 6.6</u>

(i) $\lim_{\to 0^+} \tilde{Z}(0,v) = c$ where $0 < c < \infty$.

(ii) $\int_0^1 \dfrac{dv}{\tilde{Z}(0,v)} = \infty$.

<u>Proof</u>

(i) Let $b_i = b_i(0)$. Since $W_0(x) = O(x_1^{-b_1})$ as $x_1 \to 0$ when $x_3 \geq a > 0$, there exists a neighbourhood N of $\overline{Q_0X_3}$ such that

$W_0(x) \leq k\, x_1^{-b_1}$ for $\forall\, x \in N$ with k a positive constant (Figure 6.3). But the time of passing through N for a closed orbit which intersects $\overline{EF}$ at a point with coordinate x_1 is $O(\log x_1)$ as $x_1 \to 0$. Hence in the integral $\tilde{Z}(0,v) = \int_0^{\tau_0(v)} W_\theta(x)dt$ the contribution of the region N is of the order $o(v)$ as $v \to 0$ (i.e. $x_1 \to 0$). Similarly the contribution of a neighbourhood N' of $\overline{M_0X_3}$ is of the order $o(v)$ as $v \to 0$. If the closed orbit intersects the boundaries of N and N' near Q_0 and M_0 respectively at A and B, then the contribution of the integral between A and B tends to a positive number c as $v \to 0$, because the time of flow between A and B tends to a positive limit and $W_0(x)$ is bounded above and bounded below by a positive constant in this region. This proves the first part.

Figure 6.3

(ii) We claim that $Z(0,v) = O(1-v)$ as $v \to 1^-$ from which the result will follow. As in proposition 6.1(ii), it is sufficient to show that $W_0(x) \leq L(1-V_0(x))$ for x in a neighbourhood of E and L a positive constant. Since our Lyapunov function is in algebraic form the inverse of the Lyapunov function in that proposition, we can immediately write the first terms in the Taylor expansion of V_0 about E. Putting

$x_i = \frac{1}{3} + \bar{x}_i$ as before we get:

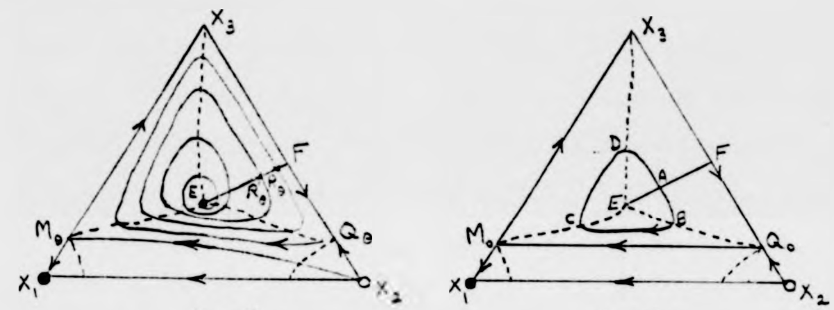
$$1-V_0(x) = -V_0(x)(1-\frac{1}{V_0(x)}) = \bar{x}_1^2[(b_1-b_3)^2-(b_1+b_3)]+2\bar{x}_1\bar{x}_2[2(b_1+b_2)^2$$
$$-2(b_1+b_2)+b_1b_2] + \bar{x}_2^2[(b_2-b_3)^2 - (b_2+b_3)] + \text{h.o.t.}$$

The coefficients of $\bar{x}_1^2$ and $\bar{x}_2^2$ are both positive since for example $(b_1-b_3)^2 - (b_1+b_3) = -4b_1(1-b_1-b_2) - b_2(1-b_2) > 0$ (remember that $b_1, b_2 < 0$, $b_3 > 0$ and $\Sigma\, b_i = 1$) . The discriminant of the quadratic is given given as in the above mentioned proposition by $\Delta = -9b_1b_2b_3 < 0$. Therefore we have exactly the conditions as in proposition 6.1 and the result follows. □

Let $(0,\bar{f})$ be another transversal deformation with $\bar{f}(0) \in I_2$. We will now construct a conjugacy between $\tilde{P}_\theta$ and $\tilde{\bar{P}}_\theta$ $(\theta \in U^* = (-\delta,\delta))$. For $\theta \neq 0$, let $h_\theta : [0,\ell_\theta] \to [0,\bar{\ell}_\theta]$ be the linear map and extend it by the relation $h_\theta \circ \tilde{P}_\theta = \tilde{\bar{P}} \circ h_\theta$ to a map $h_\theta : [0,1] \to [0,1]$ with $h_\theta(1) = 1$. Define $h_0:[0,1] \to [0,1]$ by $\int_0^{h_0(v)} \frac{dv}{\tilde{\bar{Z}}(0,v)} = \int_0^v \frac{dv}{\tilde{Z}(0,v)}$.

Then by theorem 6.2 and proposition 6.6, h_θ will induce a conjugacy between $\tilde{P}_\theta$ and $\tilde{\bar{P}}_\theta$ depending continuously on θ . It remains to extend h_θ to a map $H_\theta:\Delta \to \Delta$ inducing equivalence between (U^*,f) and $(U^*,\bar{f})$. First note that by the stable manifold theorem (proposition 4.3), the inset of M_θ and the outset of Q_θ contain segments which are arbitrary C^∞-close to the saddle connection Q_0M_0

for all sufficiently small θ . Because of the presence of saddles M_θ , Q_θ and X_3 , the construction of H_θ involves R-curves through these saddles. (Figure 6.3) We will now explain these R-curves for $\theta > 0$. Consider the region bounded by $\overrightarrow{Q_\theta R_\theta}$, $\overrightarrow{X_2M_\theta}$, $\overrightarrow{M_\theta X_3}$, $\overrightarrow{X_3F}$ and $\overline{FR_\theta}$.



$\theta > 0$ $\theta = 0$

Figure 6.3

Taking X_2 as the base of this region, we construct the R-curves through the saddles Q_θ , H_θ and X_3 . These are then extended in the region bounded by $\overrightarrow{FQ_\theta}$, $\overrightarrow{Q_\theta R_\theta}$, $\overrightarrow{R_\theta R'_\theta}$, $\overline{FR_\theta}$ and $\overline{R_\theta R'_\theta}$, where R'_θ is the image of R_θ under the return map P_θ . We continue this process until the R-curves are joined at E . In the fundamental domain bounded by $\overrightarrow{X_2M_\theta}$, $\overrightarrow{X_2X_1}$ and $\overrightarrow{M_0X_1}$ we extend the R-curve from Q_θ and construct the R-curve through M_θ . As $\theta \to 0^+$, these R-curves tend continuously to the R-curves in left picture of Figure 6.3, where in the triangle $X_3Q_0M_0$, the R-curves intersect each closed orbit such that $\ell(\overrightarrow{AB}) : \ell(\overrightarrow{BC}) : \ell(\overrightarrow{CD}) : \ell(\overrightarrow{DA}) = \ell(\overrightarrow{FQ_0}) : \ell(\overrightarrow{Q_0M_0}) : \ell(\overrightarrow{M_0X_3}) : \ell(\overrightarrow{X_3F})$. A similar construction for $\theta < 0$ leads

to R-curves which tend again to the above R-curves at $\theta = 0$.
We can now extend h_θ to $H_\theta : \Delta \to \Delta$ as follows. An orbit through a point on $\overline{EF}$ is sent to the orbit through the image of that point under h_θ such that orbit segments between the three R-curves joining at E and the line $\overline{EF}$ are mapped to the corresponding segments by fraction of arc length. The F.D.'s $X_2M_\theta X_1$ $(\theta > 0)$, $X_2Q_\theta X_1$ $(\theta < 0)$ and $X_2Q_0M_0X_1$ $(\theta = 0)$ are mapped to their corresponding domains as in chapter 4. Finally all orbits on $\partial\Delta$ are sent to the corresponding orbits by fraction of length. This completes the construction of H_θ and we have therefore proved: □

Theorem 6.7

$(0,f)$ and $(0,\bar{f})$ are equivalent.

Corollary 6.8

A transversal deformation of $A \in I_2$ is topologically versal and A has "codimension" one. □

6.3 Deformations of I_3

Let $(0,f)$ be a transversal deformation of $f(0) \in I_3$ with (U,f) a representative unfolding. We can assume WLG that (U,f) is central and of the form:

$$f(\varepsilon) = \begin{pmatrix} 0 & \theta(\varepsilon)+a_1(\varepsilon) & \theta(\varepsilon)-a_1(\varepsilon) \\ \theta(\varepsilon)-a_2(\varepsilon) & 0 & \theta(\varepsilon)+a_2(\varepsilon) \\ \theta(\varepsilon)+a_3(\varepsilon) & \theta(\varepsilon)-a_3(\varepsilon) & 0 \end{pmatrix} , \quad \varepsilon \in U$$

with $\begin{cases} 0 < a_2(\varepsilon) < a_1(\varepsilon)\ ,\ a_3(\varepsilon) \\ \\ 0 < \theta(\varepsilon) < a_1(\varepsilon)\ ,\ a_3(\varepsilon) \end{cases}$ for $\forall\ \varepsilon \in U$, $a_2(\varepsilon) \gtrless \theta(\varepsilon)$ for $\varepsilon \gtrless 0$,

and the transversality condition $a_2'(0) - \theta'(0) > 0$.

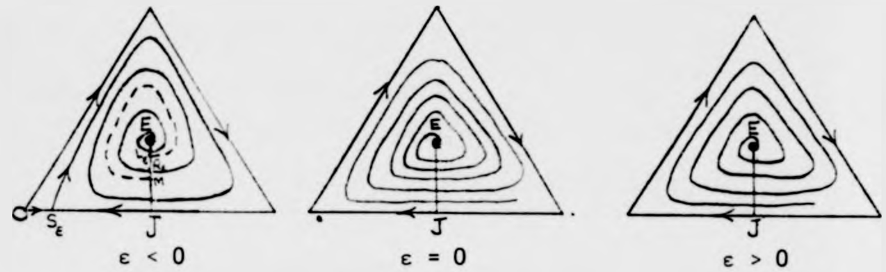


Figure 6.4

Observe that $V_\varepsilon(x) = \dfrac{\prod_i x_i^{b_i(\varepsilon)}}{\sum_i b_i(\varepsilon)x_i}$ is a Lyapunov function with

$$V_\varepsilon(x) = \theta(\varepsilon)\ \frac{\prod_i x_i^{b_i(\varepsilon)}\ \Sigma\ b_i(\varepsilon)b_j(\varepsilon)\ (x_i - x_j)^2}{(\Sigma\ b_i(\varepsilon)x_i)^2} > 0$$

for $\forall\ \varepsilon \in U$. We also note that the eigenvalues at E , $-\frac{\theta}{3} \pm \sqrt{-\rho}$, are complex conjugate for $\forall\ \varepsilon \in U$. Furthermore, lemma 6.5 holds in this case as well since $a_i(\varepsilon) + a_{i+1}(\varepsilon) \geq 0$, $i = 1,2,3$, $\forall\ \varepsilon \in U$. Therefore all orbits in $\overset{\circ}{\Delta}\setminus E$ intersect $\overline{EJ}$ transversally (J is the midpoint of $\overline{X_1X_2}$) .

The idea for using the function T below is due to Zeeman. Let

S_ε denote the hyperbolic saddle on $\partial\Delta$ $(\varepsilon < 0)$. The forward orbit through the midpoint M of $\overline{EJ}$ first returns back to $\overline{EJ}$ at L_ε (Figure 6.4). $W^u(S_\varepsilon)$, the outset of S_ε , intersects the half open segment $\overline{ML_\varepsilon}$ (open from L_ε) at a unique point R_θ . Let $N(\varepsilon)$ be the number of intersections of $W^u(S_\varepsilon)$ with $\underset{-}{\overset{\Omega}{JM}}$ and define $T(\varepsilon) = N(\varepsilon) + \dfrac{\ell(\overline{R_\varepsilon L_\varepsilon})}{\ell(\overline{ML})}$.

Proposition 6.9

(i) T depends continuously on ε and $T(\varepsilon) \to \infty$ as $\varepsilon \to 0^-$.

(ii) If $a_i(\varepsilon) = a_i$ is constant $(i = 1,2,3)$, then $\theta_1 > \theta_2 \Rightarrow T(\theta_1) < T(\theta_2)$.

Proof

The continuity of T is an immediate consequence of the fact that the i^{th} intersection of $W^u(S_\varepsilon)$ with $\overline{EJ}$ (counted from J) depends continuously on ε , which itself follows from the stable manifold theorem (proposition 4.3). To prove that $T(\varepsilon) \to \infty$ as $\varepsilon \to 0^-$, choose arbitrary large positive integer n . At $\varepsilon = 0$, there is a cycle of saddles at the vertices of Δ and hence the backward orbit of M intersects $\overline{MJ}$ infinitely often. Pick a point A on $\overline{MJ}$ such that $= \{\sigma^-_{\Delta_{f(0)}}(M) \cap \overline{MA}\} > n$. Let P_ε denote the return map on $\overline{EJ}$ for the flow $\Delta_{f(\varepsilon)}$ and put $Q_0 = P_0^{-n}(M)$. By the continuous dependence of the n^{th} iteration of the inverse of the return map with respect to the parameter we conclude that for small ε , $Q_\varepsilon = P_\varepsilon^{-n}(M)$ exists (and is near Q_0) . Hence $T(\varepsilon) > n$ for all sufficiently small ε , i.e. $f(\varepsilon) \to \infty$ as $\varepsilon \to 0^-$.

(ii) Since a_i's are constant and $\left.\frac{d\theta}{d\varepsilon}\right|_{\varepsilon=0} \neq 0$ we can take θ as the parameter, so that T becomes a function of θ . The vector field is:

$$\dot{x}_i = V_i^\theta(x) = x_i[a_i x_{i+1} - a_i x_{i+2} - \sum_j x_j x_{j+1}(a_j - a_{j+1})$$
$$+ \theta(x_{i+1} + x_{i+2}) - 2\theta \sum_{j<k} x_j x_k] \ .$$

Let $\theta_1 > \theta_2$. We will calculate the cross product of the two vector fields at θ_1 and θ_2 . We have:

$$V^{\theta_1}(x) \wedge V^{\theta_2}(x) = \ell(\theta_1,\theta_2,x)(1,1,1,)$$

where $\ell(\theta_1,\theta_2,x) = V_1^{\theta_1}(x)V_2^{\theta_2}(x) - V_1^{\theta_2}(x)V_2^{\theta_1}(x)$. Writing $x_3 = 1-x_1-x_2$ we find after a long calculation in appendix 3(b) that:

$$\ell(\theta_1,\theta_2,x) = (\theta_1-\theta_2)x_1x_2\{(a_1+a_2)-(3a_1+5a_2)x_1-(5a_1+3a_2)x_2$$
$$+ (3a_1+8a_2+a_3)x_1^2+(8a_1+3a_2+a_3)x_2^2 + 2(5a_1+5a_2-a_3)x_1x_2$$
$$- (a_1+4a_2+a_3)x_1^3 - (4a_1+a_2+a_3)x_2^3 + (a_3-5a_1-8a_2)x_1^2x_2 + (a_3-8a_1-5a_2)x_1x_2^2\} \ .$$

By cyclic symmetry $x_3 = 1-x_1-x_2$ must be a factor of $\ell(\theta_1,\theta_2,x)$ and in fact on dividing by $1-x_1-x_2$ we find that:

$$\ell(\theta_1,\theta_2,x) = (\theta_1-\theta_2)x_1x_2(1-x_1-x_2)\{(a_1+4a_2+a_3)x_1^2 + (4a_1+a_2+a_3)x_2^2$$
$$+ 2(2a_1+2a_2-a_3)x_1x_2 - 2(a_1+2a_2)x_1-2(2a_1+a_2)x_2+(a_1+a_2)\} \ .$$

The quadratic in the bracket has a stationary point at $E = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, where it vanishes. The discriminant is given by

$$\Delta = (2a_1+2a_2-a_3)^2 - (a_1+4a_2+a_3)(4a_1+a_2+a_3) =$$
$$-9 \sum_{i<j} a_i a_j < 0 \quad .$$

Hence the quadratic takes its global minimum value at E and therefore $\ell(\theta_1,\theta_2,x) > 0$ for $x \in \mathring{\Delta}\backslash E$. By proposition 1.3(vii), S_{θ_2} is between X_1 and S_{θ_1} (Figure 6.5). Therefore to prove that $T(\theta_1) < T(\theta_2)$ it is sufficient to show that the outset of S_{θ_2} (for the flow $\Delta_{f(\theta_2)}$) does not intersect the outset of S_{θ_1} (for the flow $\Delta_{f(\theta_1)}$). But if these outsets do intersect then at the point of intersection we must have $\ell(\theta_1,\theta_2,x) \leq 0$ which contradicts the above result. The proof then follows. □


Figure 6.5

If a_i's are not constant then the conclusion of the second part of the proposition may not hold. Consequently we can only prove a less general version of the results that we have obtained for all other I_i's. Let $(0,f),(0,\bar{f})$ be transversal deformations of $f(0)$, $\bar{f}(0) \in I_3$ with constant a_i's and $\bar{a}_i$'s. We will show that these two deformations are equivalent, but, unlike all other cases, this time a nontrivial reparametrization is necessary.

Theorem 6.10

(O,f) and $O,\bar{f})$ are equivalent.

Proof

We take θ as the parameter and consider representative unfoldings (U^*,f), $(\bar{U}^*,\bar{f})$ with $U^* = (a_2-\delta, a_2+\delta)$, $\bar{U}^* = (\bar{a}_2-\delta, \bar{a}_2+\delta')$ where $0 < \delta << \min(a_1,a_3)$ and δ' is given by $\bar{a}_2+\delta' = \bar{T}^{-1}T(a_2+\delta)$.

Define $\eta : U^* \to \bar{U}^*$ by $\eta(\theta) = \begin{cases} \theta-a_2+\bar{a}_2 & a_2-\delta < \theta \le a_2 \\ \bar{T}^{-1}T(\theta) & a_2 < \theta < a_2+\delta \end{cases}$.

Since by the last proposition T and $\bar{T}$ are strictly decreasing in $(a_2,a_2+\delta)$ and $(\bar{a}_2,a_2+\delta')$ respectively, $\bar{T}^{-1}T$ is a well defined continuous function with inverse in $(a_2,a_2+\delta)$. Also $\bar{T}^{-1}T(\theta) \to \bar{a}_2^+$ as $\theta \to a_2^+$. Hence η is a homeomorphism of U^* onto $\bar{U}^*$. Now for each $\theta \in U^*$ we define a homeomorphism h_θ of $\overline{EJ}$ onto itself which induces a conjugacy between P_θ and $\bar{P}_{\eta(\theta)}$ (the return maps on $\overline{EJ}$ for the flows $\Delta_{f(\theta)}$ and $\Delta_{\bar{f}(\eta(\theta))}$), which depends continuously on θ. We let h_θ map $\overline{ML}_\theta$ onto $\overline{M\bar{L}}_{\eta(\theta)}$ linearly and extend it by the relation $h_\theta \circ P_\theta = \bar{P}_{\eta(\theta)} \circ h_\theta$ to $\overline{EJ}$ with E and J mapped to themselves. Notice that since by the definition of $\eta_{(\theta)}$, for

$a_2 < \theta < a_2 + \delta$, $\dfrac{\ell(\overline{R_\theta L_\theta})}{\ell(\overline{ML_\theta})} = \dfrac{\ell(\overline{\bar{R}_{\eta(\theta)}\bar{L}_{\eta(\theta)}})}{\ell(\overline{M\bar{L}_{\eta(\theta)}})}$, R_θ is mapped to $\bar{R}_{\eta(\theta)}$

and hence the i^{th} intersection of $W^u(S_\theta)$ with $\overline{EJ}$ is mapped to the i^{th} intersection of $W^u(\bar{S}_{\eta(\theta)})$ with $\overline{EJ}$ for any integer i.

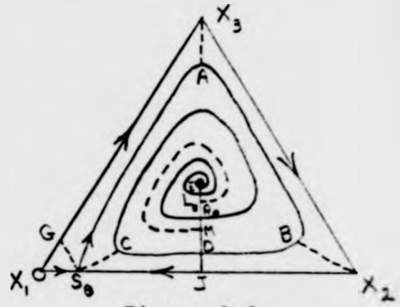


Figure 6.6

Note also that for $\forall\ \theta \in U^*$, $\overline{ML_\theta}$ and $\overline{ML_{n(\theta)}}$ are nonvanishing intervals and therefore our construction of h_θ makes sense. It remains to extend h_θ to a map $H_\theta : \Delta \to \Delta$. For $a_2 - \delta < \theta \le a_2$, the construction of H_θ is as in theorem 6.3, i.e. an orbit segment from $x \in \overline{EJ}$ to $P_\theta(x)$ is mapped onto the corresponding orbit segment from $h_\theta(x)$ to $\bar{P}_{n(\theta)} \circ h_\theta(x)$ by fraction of arc length, and $H_\theta|_{\partial\Delta}$ = Identity. For $a_2 < \theta < a_2 + \delta'$, we need R-curves through the saddles. Let $W^u(S_\theta)$ intersect $\overline{EJ}$ first at D and take points A,B and C on $W^u(S_\theta)$ such that $\ell(\overrightarrow{S_\theta A}) : \ell(\overrightarrow{AB}) : \ell(\overrightarrow{BD}) : \ell(\overrightarrow{DC}) = 1 : 1 : \frac{1}{2} : \frac{1}{2}$. Let $G \in \overline{X_1X_2}$ be such that $\ell(\overrightarrow{X_1G}) = \ell(\overrightarrow{X_1S_\theta})$. Then take the R-curves between S_θ and G, X_3 and A, X_2 and B, and finally between S_θ and C, all with respect to the domain bounded by $\overrightarrow{X_1X_3}$, $\overrightarrow{X_3X_2}$, $\overrightarrow{X_2J}$, $\overrightarrow{S_\theta D}$ and $\overline{JD}$. Now construct H_θ by sending the orbit segment from $x \in \overline{EJ}\backslash J$ to $P_\theta(x)$ onto the orbit segment from $h_\theta(x)$ to $\bar{P}_{n(\theta)} \circ h_\theta(x)$, and the backward orbit of $x \in \overline{DJ}\backslash J$ to the backward orbit of $h_\theta(x)$, such that segments between the R-curves and $\overline{ED}$ are mapped onto the corresponding segments by fraction of arc length. All orbits on $\partial\Delta$ are mapped to their corresponding orbits by fraction of length. It is then easy to check that for each $\theta \in (a_2, a_2+\delta)$, H_θ is a homeomorphism and that H_θ depends continuously on θ as $\theta \to a_2^+$. This completes the construction of H_θ inducing an equivalence between (U^*, f) and $(\bar{U}^*, \bar{f})$. □

We can still deduce:

Corollary 6.11

Any matrix $A \in I_3$ has "codimension" one.

Proof

WLG assume A is central. Let $(0,f)_k$ be any k-deformation of A and let $(0,\bar{f})$ be the transversal deformation of A which is central with constant a_i's as in the theorem. Put $j = \bar{T}^{-1}T$, where T and $\bar{T}$ are the germs of the functions introduced in this section. Then $(0,\bar{f}\circ j)_k$ is a deformation induced from $(0,\bar{f})$ and is equivalent to $(0,f)_k$. Hence $(0,\bar{f})$ is topologically versal and the result follows. □

We have therefore proved in the course of three chapters that any cod 1 matrix $A \in I_i$ $(1 \le i \le 38)$ has in fact "codimension" one, which completes our study of the cod 1 bifurcations. Our last remark is that the homeomorphism H_ε, for fixed $\varepsilon \neq 0$, in theorems 4.5 6.3, 6.7 and 6.10 induces an equivalence between any two matrices in the same cod 0 stratum. We have therefore established an alternative and simpler proof of the main result in Carvalho's thesis.

Chapter 7.

Codimension two Bifurcations

In this final chapter we will determine all the cod 2 strata of different types together with their bifurcation diagrams. We will also point out the shortcomings and errors of Bomze in his attempt to find all the possible phase portraits of the flows in the planar replicator equations.

If $A \in Z_3$ is a cod 2 matrix, then at most two of its off-diagonal entries can vanish. Therefore there are two possible cases:

(i) In each column of A at most one off-diagonal element is zero. In this case, A or its equivalent can be obtained from $\operatorname{Im} W$ or Q (see section 2.3). Therefore such matrices are represented by the cod 2 strata in Q and by using Figures 2.3 and 2.4 we find that there are 35 cod 2 strata in Q which we denote by Π_i $(1 \le i \le 35)$. The complete list of these with an example of a transversal deformation for each is given in appendix 2. The full bifurcation diagram in the case of $A \in \Pi_3$ is sketched in Figure 7.1.

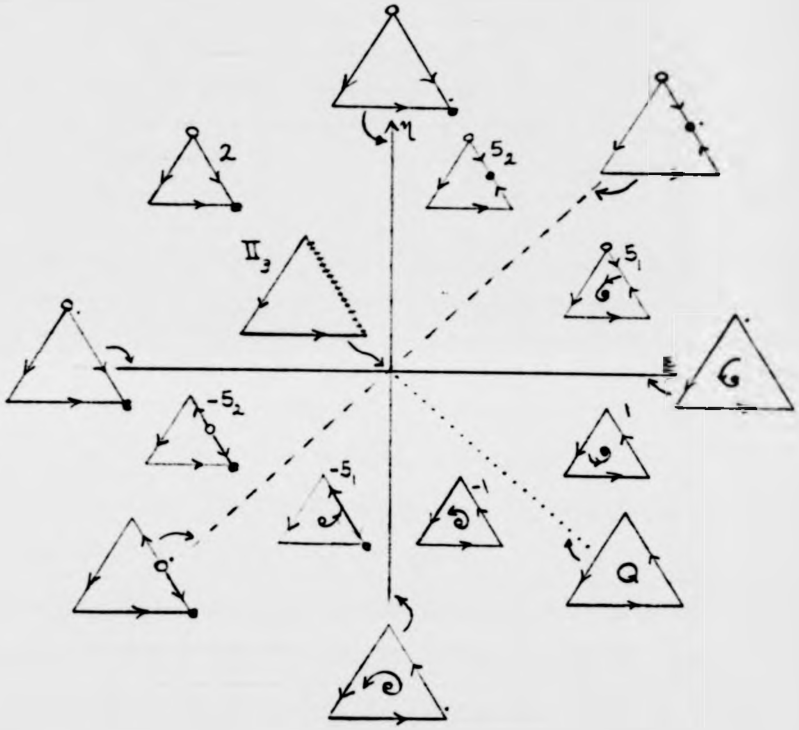
(ii) In one column of A both off-diagonal entries are zero. By cyclic symmetry we can assume the second column has zero entries. Let $\tilde{Q} = (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^2 \times (\mathbb{R}/2\pi\mathbb{Z} = S^1 \times \mathbb{R}^2 \times S^1$ and for $\alpha_1, \alpha_3 \in S^1$, Bifurcation of $f(\varepsilon, \eta) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & \eta \\ -1 & \varepsilon & 0 \end{pmatrix}$

$f(0,0) \in \Pi_3$ $(\alpha_1 = \frac{\pi}{6}, \alpha_3 = \frac{\pi}{3})$

o = attractor

o = repellor

. = zero eigenvalue



dashed line = line is pointwise fixed

solid line = {Exchange of stability at a vertex

broken line = {Exchange of stability in the interior of an edge

dotted line = {Degenerate Hopf bifurcation in the interior of Δ

Figure 7.1

$(\varepsilon,\eta) \in \mathbb{R}^2$ define the map

$$\begin{cases} \tilde{W} : \tilde{Q} \to Z_3 \\ \tilde{\alpha} \to \begin{pmatrix} 0 & \varepsilon & \mathrm{Sin}\alpha_3 \\ \mathrm{Sin}\alpha_1 & 0 & \mathrm{Sin}(\alpha_3 - \frac{\pi}{3}) \\ \mathrm{Sin}(\alpha_1 - \frac{\pi}{3}) & \eta & 0 \end{pmatrix}, \quad \tilde{\alpha} = (\alpha_1,\varepsilon,\eta,\alpha_3) \in \tilde{Q} \ . \end{cases}$$

Then A is equivalent to a matrix of the form $\tilde{W}(\alpha_1,0,0,\alpha_3)$ and hence belongs to one of the cod 2 strata of $\mathrm{Im}\,\tilde{W}$ or equivalently $\tilde{Q}$ with $\varepsilon = \eta = 0$. We can find these strata in $\tilde{Q}$ and the bifurcation diagram corresponding to a transversal deformation of a matrix in each of them by putting $\varepsilon = r\,\mathrm{Sin}(\alpha_2 - \frac{\pi}{3})$, $\eta = r\,\mathrm{Sin}\alpha_2$ and using Q instead. We find that there are 18 different strata as such, which are listed as Π_i $(36 \le i \le 53)$. As one can see in appendix 2, some of these strata induce the same phase portrait although they are clearly of different types since they are surrounded by different sets of stable classes. Figure 7.2 presents the bifurcation diagram for one of these strata.

We have therefore obtained the following result.

<u>Proposition 7.1</u>

Up to time reversal, there are 53 cod 2 strata of different types. □

Bifurcation of $f(\varepsilon,\eta) = \begin{pmatrix} 0 & -\varepsilon & 1 \\ 1 & 0 & 2 \\ 2 & \eta & 0 \end{pmatrix}$

$f(0,0) \in \Pi_{47}$ $(\alpha_1 = \frac{5\pi}{6}, \alpha_3 = \frac{5\pi}{6})$

Notations as in Figure 7.1

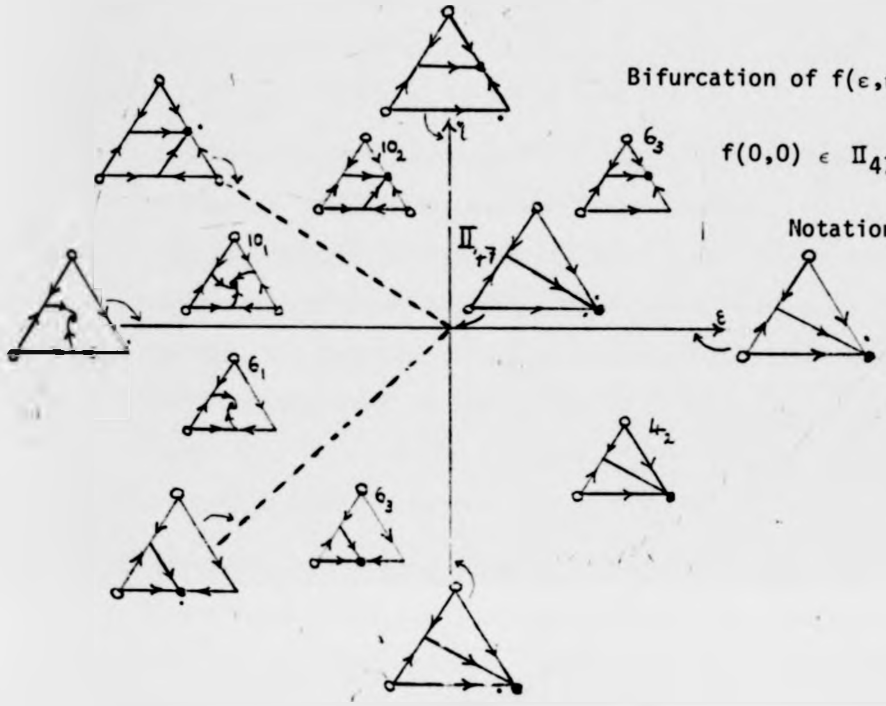
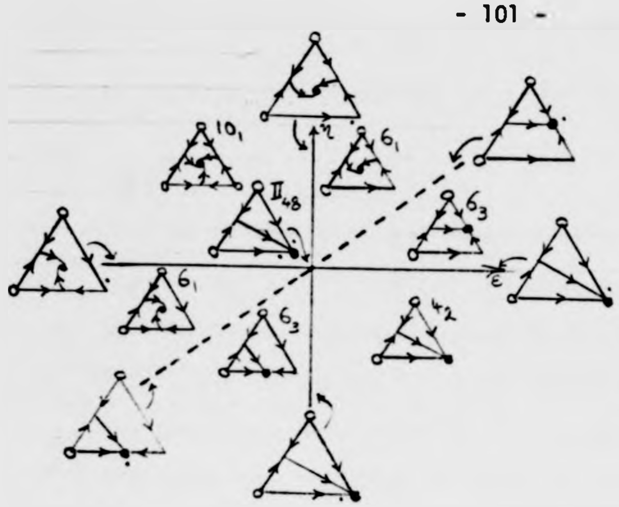


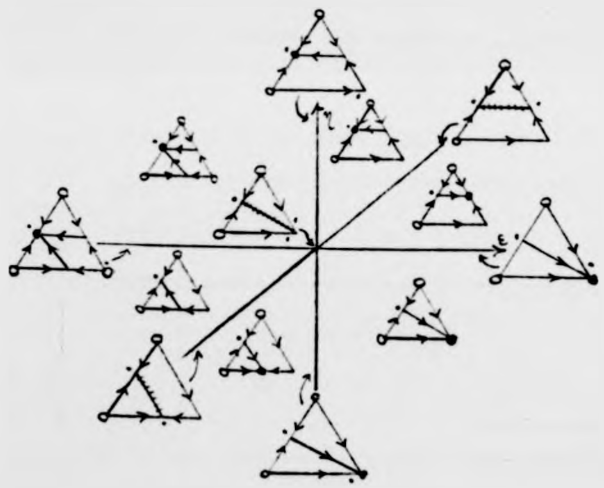
Figure 7.2

Our conjecture is that transversal deformations of cod 2 matrices are topologically versal and hence all these matrices have "codimension two". However, except in the simplest cases, it is a formidable task to prove this. As a codimension two problem, it has much less practical importance and we will not pursue it any further.

Our last remark concerns Bomze's results [7]. He has used an ad hoc method to try to determine all the possible phase portraits of the planar replicator system by classifying the flows according to the number of fixed points in $\mathring{\Delta}$ or on $\partial\Delta$. Consequently, he presents 47 phase portraits which is supposed to be a complete list of all possible phase portraits. However this result is totally inadequate. Firstly, it is not based on the notion of stability of the flows and hence flows of different codimensions are confused. Secondly, no bifurcation diagram can be produced by this method and therefore one cannot see how these phase portraits are related. Thirdly, the method is cumbersome and tedious and contains errors to the effect that his list is at least short of one phase portrait. In fact, he claims on the basis of a false proof that if a line is pointwise fixed in $\overset{\circ}{\Delta}$, then it cannot go through a vertex. However the matrix $A = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}$ induces precisely such a flow. Indeed $\frac{1}{2}A = \tilde{W}(\frac{5\pi}{6}, 0, 0, \frac{\pi}{2})$ and it is easy to check from Figure 2.4 (cross-section on the right) with $\alpha_1 = \frac{5\pi}{6}$ and $\alpha_3 = \frac{\pi}{2}$ that A is a cod 3 matrix. The three parameter family $\begin{pmatrix} 0 & \varepsilon & 2 \\ 1 & 0 & 1+\delta \\ 2 & \eta & 0 \end{pmatrix}$ is a transversal deformation of A . One can also check that the vector field $\dot{x} = V^A(x)$ vanishes on the line $x_1 = x_3$ i.e. this line is pointwise fixed. In Figure 7.3 we have sketched the bifurcation diagrams for the three sections $\delta > 0$, $\delta = 0$ and $\delta > 0$.

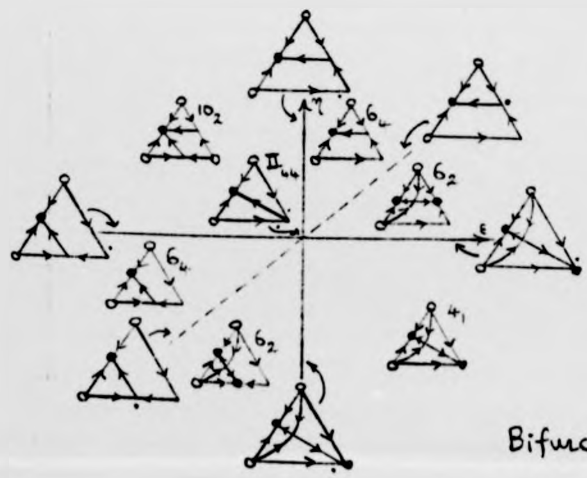


$\delta > 0$
(Bifurcation of II_{48} in appendix 2)



$\delta = 0$

Figure 7.3

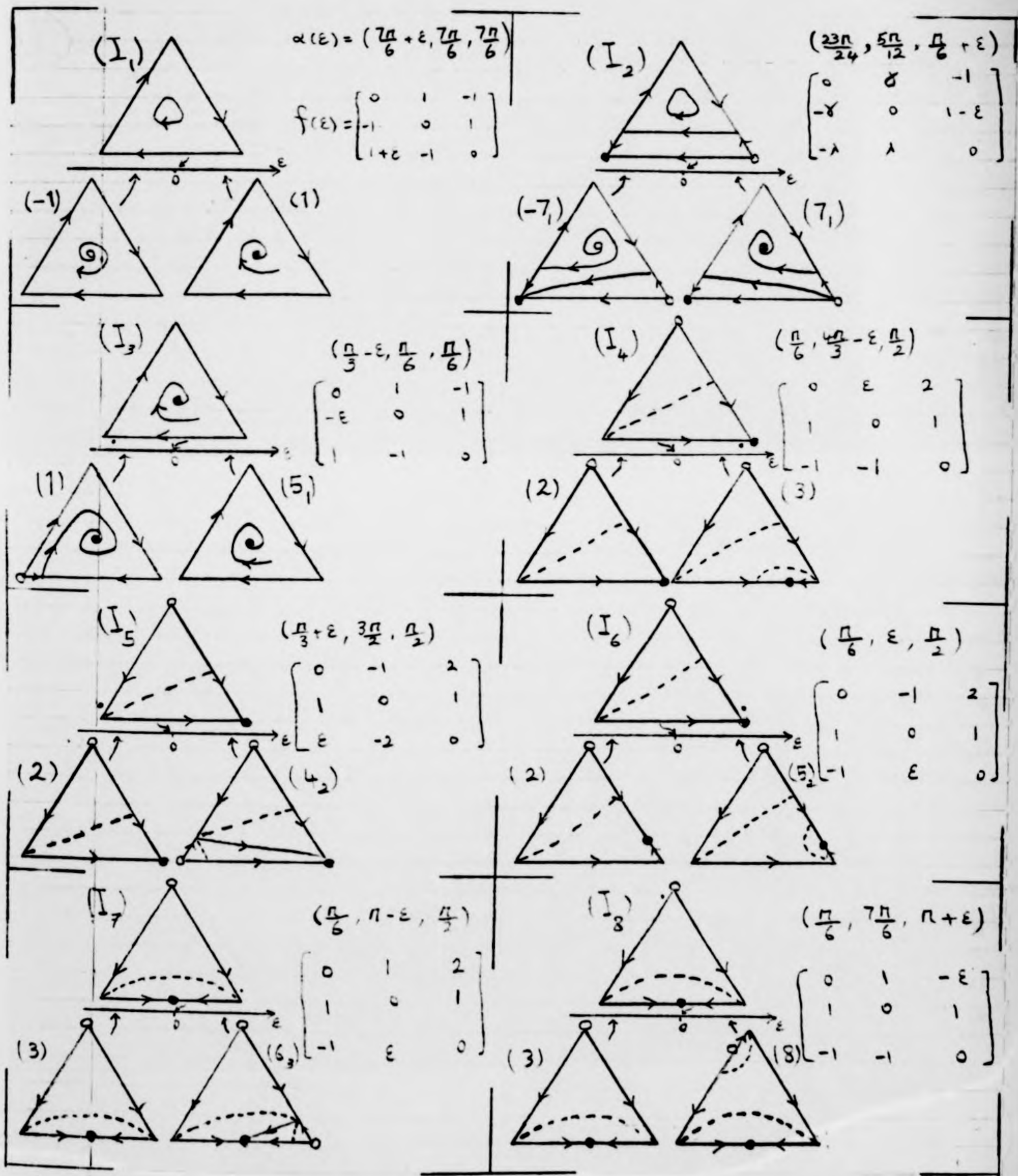


$\delta < 0$
(Bifurcation of II_{44} in appendix 2)

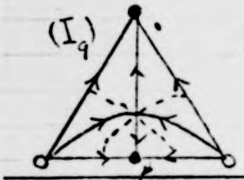
Bifurcation of $f(\varepsilon,\eta,\delta) = \begin{bmatrix} 0 & -\varepsilon & 2 \\ 1 & 0 & 1+\delta \\ 2 & \eta & 0 \end{bmatrix}$

Appendix 1

We present here an example of a matrix in each of the cod 1 strata I_i $(1 \le i \le 38)$, together with a transversal deformation and the corresponding bifurcation diagram. For each diagram, the flow above the arrowed line is induced by the cod 1 matrix whereas the two flows underneath are the neighbouring stable flows. The arrow indicates the direction of positive values of the parameter ε. A small dot near a vertex and facing an edge means that the eigenvalue at that vertex corresponding to the eigenvector in the direction of that edge is zero. Similarly a small dot near a fixed point in the interior of an edge implies that the eigenvalue corresponding to the eigenvector transversal to that edge is zero. In the bifurcation diagrams for I_i $(4 \le i \le 38)$, the broken curves are the R-curves determined by section 4.1. Finally we have used the notations $\lambda = \mathrm{Sin}\,\frac{\pi}{6}$ and $\gamma = \mathrm{Sin}\,\frac{5\pi}{6}$.



(I_1)
$\alpha(\varepsilon) = (\frac{7\pi}{6}+\varepsilon, \frac{7\pi}{6}, \frac{7\pi}{6})$
$f(\varepsilon) = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1+\varepsilon & -1 & 0 \end{bmatrix}$
ε
0
(-1)
(1)
(I_2)
$(\frac{23\pi}{24}, \frac{5\pi}{12}, \frac{\pi}{6}+\varepsilon)$
$\begin{bmatrix} 0 & \gamma & -1 \\ -\gamma & 0 & 1-\varepsilon \\ -\lambda & \lambda & 0 \end{bmatrix}$
(-7_1)
(7_1)
(I_3)
$(\frac{\pi}{3}-\varepsilon, \frac{\pi}{6}, \frac{\pi}{6})$
$\begin{bmatrix} 0 & 1 & -1 \\ -\varepsilon & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$
(1)
(5_1)
(I_4)
$(\frac{\pi}{6}, \frac{4\pi}{3}-\varepsilon, \frac{\pi}{2})$
$\begin{bmatrix} 0 & \varepsilon & 2 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$
(2)
(3)
(I_5)
$(\frac{\pi}{3}+\varepsilon, \frac{3\pi}{2}, \frac{\pi}{2})$
$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ \varepsilon & -2 & 0 \end{bmatrix}$
(2)
(4_2)
(I_6)
$(\frac{\pi}{6}, \varepsilon, \frac{\pi}{2})$
$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ -1 & \varepsilon & 0 \end{bmatrix}$
(2)
(5_2)
(I_7)
$(\frac{\pi}{6}, \pi-\varepsilon, \frac{\pi}{2})$
$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ -1 & \varepsilon & 0 \end{bmatrix}$
(3)
(6_3)
(I_8)
$(\frac{\pi}{6}, \frac{7\pi}{6}, \pi+\varepsilon)$
$\begin{bmatrix} 0 & 1 & -\varepsilon \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$
(3)
(8)



$\alpha(\varepsilon) = \left(\frac{\pi}{2}, \frac{11\pi}{12}, \frac{4\pi}{3} - \varepsilon\right)$

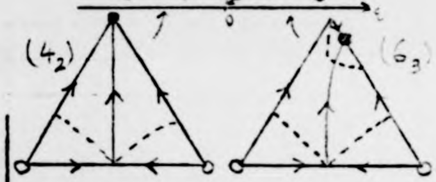
$$f(\varepsilon) = \begin{bmatrix} 0 & \gamma & -1 \\ 2 & 0 & \varepsilon \\ 1 & \lambda & 0 \end{bmatrix}$$

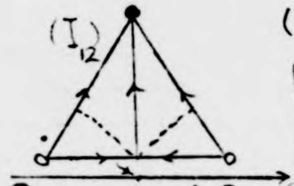
$\left(\frac{\pi}{3} - \varepsilon, \frac{5\pi}{6}, \frac{3\pi}{2}\right)$

$$\begin{bmatrix} 0 & 2 & -2 \\ 1 & 0 & -1 \\ -\varepsilon & 1 & 0 \end{bmatrix}$$

$\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{4\pi}{3} - \varepsilon\right)$

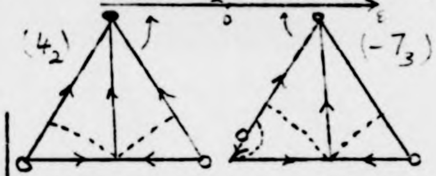
$$\begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & \varepsilon \\ 1 & 2 & 0 \end{bmatrix}$$





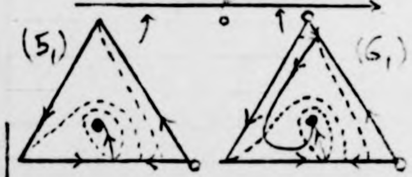
$\left(\frac{\pi}{3} - \varepsilon, \frac{\pi}{2}, \frac{3\pi}{2}\right)$

$$\begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & -1 \\ -\varepsilon & 2 & 0 \end{bmatrix}$$



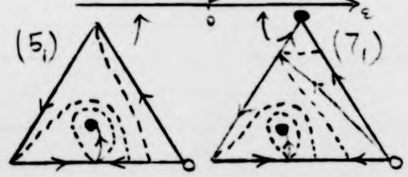
$\left(\frac{\pi}{6}, \frac{5\pi}{12}, \frac{\pi}{3} + \varepsilon\right)$

$$\begin{bmatrix} 0 & \lambda & 1 \\ 1 & 0 & \varepsilon \\ -1 & \gamma & 0 \end{bmatrix}$$



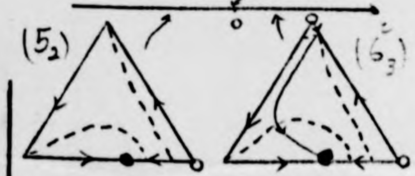
$\left(\frac{\pi}{6}, \frac{5\pi}{12}, -\varepsilon\right)$

$$\begin{bmatrix} 0 & \lambda & -\varepsilon \\ 1 & 0 & -1 \\ -1 & \gamma & 0 \end{bmatrix}$$



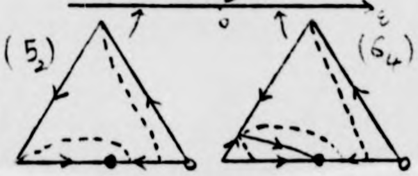
$\left(\frac{\pi}{6}, \frac{2\pi}{3}, \frac{\pi}{3} + \varepsilon\right)$

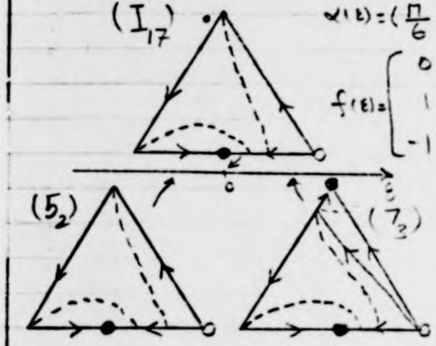
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & \varepsilon \\ -1 & 1 & 0 \end{bmatrix}$$



$\left(\frac{\pi}{3} + \varepsilon, \frac{5\pi}{6}, \frac{\pi}{6}\right)$

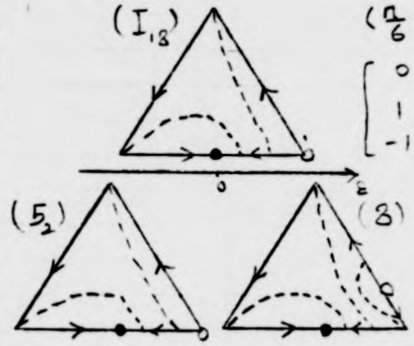
$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -1 \\ \varepsilon & 1 & 0 \end{bmatrix}$$





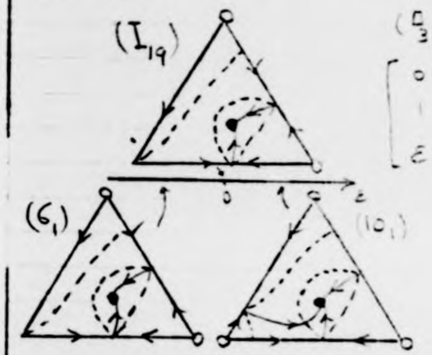
$\alpha(\varepsilon) = \left(\frac{\pi}{6},\ \frac{5\pi}{6},\ -\varepsilon\right)$

$f(\varepsilon) = \begin{bmatrix} 0 & 2 & -\varepsilon \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$



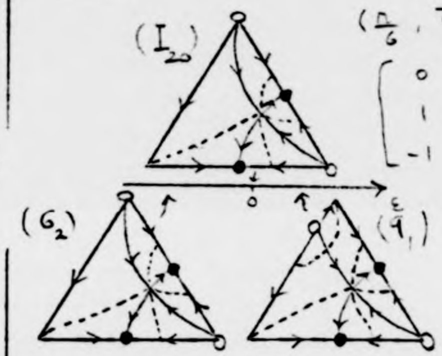
$\left(\frac{\pi}{6},\ \pi+\varepsilon,\ \frac{\pi}{6}\right)$

$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -\varepsilon & 0 \end{bmatrix}$



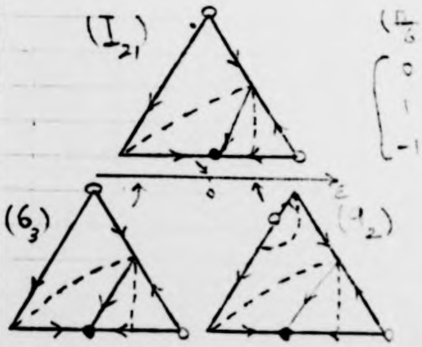
$\left(\frac{\pi}{3}+\varepsilon,\ \frac{\pi}{2},\ \frac{\pi}{2}\right)$

$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ \varepsilon & 2 & 0 \end{bmatrix}$



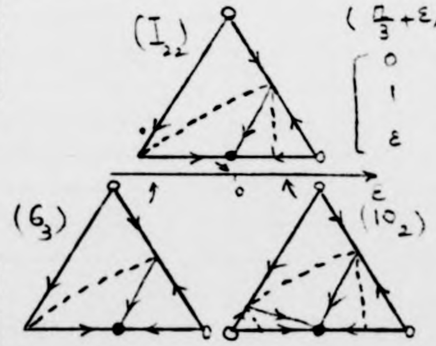
$\left(\frac{\pi}{6},\ \frac{7\pi}{12},\ \pi+\varepsilon\right)$

$\begin{bmatrix} 0 & \sqrt{2} & -\varepsilon \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$



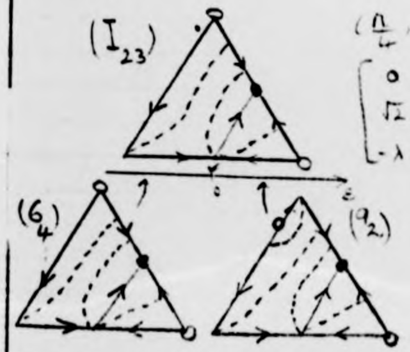
$\left(\frac{\pi}{6},\ \frac{5\pi}{6},\ \pi+\varepsilon\right)$

$\begin{bmatrix} 0 & 2 & -\varepsilon \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$



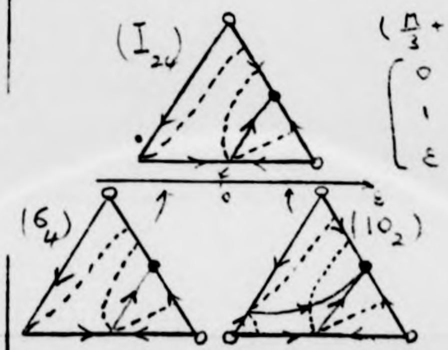
$\left(\frac{\pi}{3}+\varepsilon,\ \frac{5\pi}{6},\ \frac{\pi}{2}\right)$

$\begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & 1 \\ \varepsilon & 1 & 0 \end{bmatrix}$



$\left(\frac{\pi}{4},\ \frac{\pi}{2},\ \pi+\varepsilon\right)$

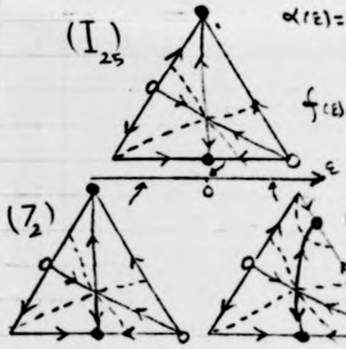
$\begin{bmatrix} 0 & 1 & -\varepsilon \\ \sqrt{2} & 0 & 1 \\ -\lambda & 2 & 0 \end{bmatrix}$



$\left(\frac{\pi}{3}+\varepsilon,\ \frac{5\pi}{12},\ \frac{5\pi}{6}\right)$

$\begin{bmatrix} 0 & \lambda & 1 \\ 1 & 0 & 2 \\ \varepsilon & \gamma & 0 \end{bmatrix}$

(I_{25})

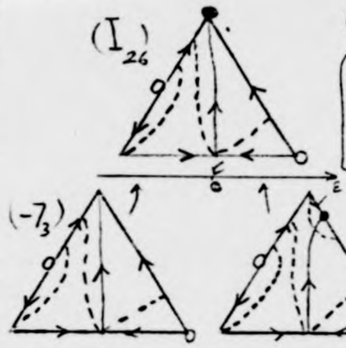


(7_2) (9_1)

$$\alpha(\varepsilon) = \left(\frac{\pi}{6}, \frac{5\pi}{6}, \frac{4\pi}{3} - \varepsilon\right)$$

$$f(\varepsilon) = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 0 & \varepsilon \\ -1 & 1 & 0 \end{bmatrix}$$

(I_{26})

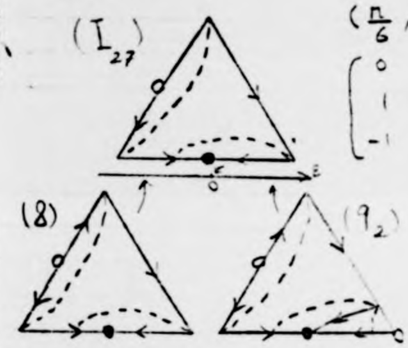


(-7_3) (9_2)

$$\left(\frac{\pi}{6}, \frac{5\pi}{12}, \frac{4\pi}{3} - \varepsilon\right)$$

$$\begin{bmatrix} 0 & \lambda & -1 \\ 1 & 0 & \varepsilon \\ -1 & \gamma & 0 \end{bmatrix}$$

(I_{27})

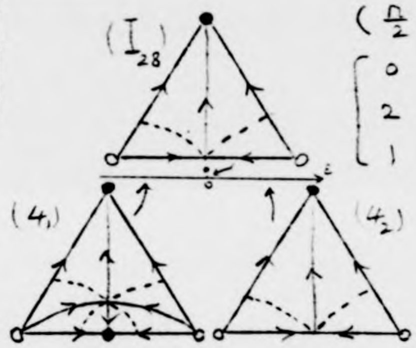


(8) (9_2)

$$\left(\frac{\pi}{6}, \pi - \varepsilon, \frac{7\pi}{6}\right)$$

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & \varepsilon & 0 \end{bmatrix}$$

(I_{28})

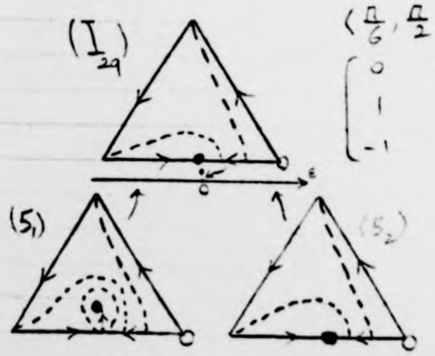


(4_1) (4_2)

$$\left(\frac{\pi}{2}, \frac{5\pi}{6} - \varepsilon, \frac{3\pi}{2}\right)$$

$$\begin{bmatrix} 0 & 2 & -2 \\ 2 & 0 & -1 \\ 1 & 1+\varepsilon & 0 \end{bmatrix}$$

(I_{29})

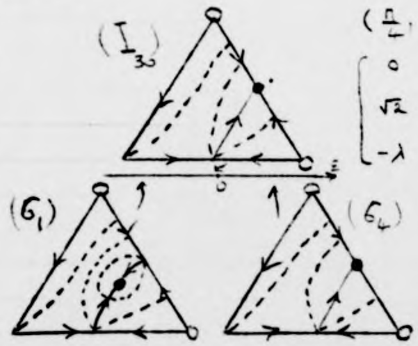


(5_1) (5_2)

$$\left(\frac{\pi}{6}, \frac{\pi}{2} + \varepsilon, \frac{\pi}{6}\right)$$

$$\begin{bmatrix} 0 & 1+\varepsilon & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 0 \end{bmatrix}$$

(I_{30})

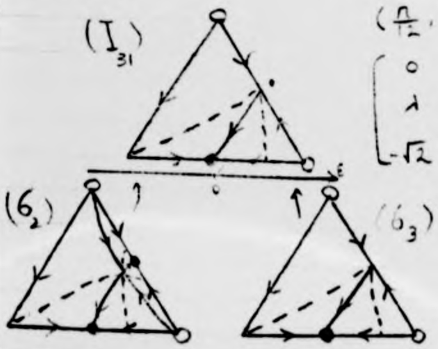


(6_1) (6_4)

$$\left(\frac{\pi}{4}, \frac{\pi}{2} - \varepsilon, \frac{5\pi}{6}\right)$$

$$\begin{bmatrix} 0 & 1-\varepsilon & 1 \\ \sqrt{2} & 0 & 2 \\ -\lambda & 2 & 0 \end{bmatrix}$$

(I_{31})

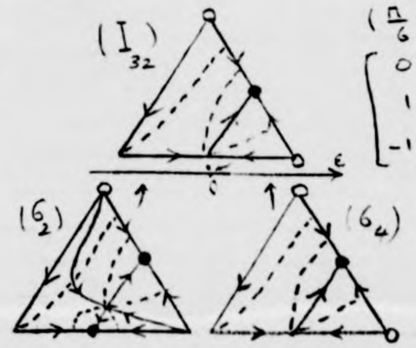


(6_2) (6_3)

$$\left(\frac{\pi}{12}, \frac{\pi}{2} + \varepsilon, \frac{5\pi}{6}\right)$$

$$\begin{bmatrix} 0 & 1+\varepsilon & 1 \\ \lambda & 0 & 2 \\ -\sqrt{2} & 2 & 0 \end{bmatrix}$$

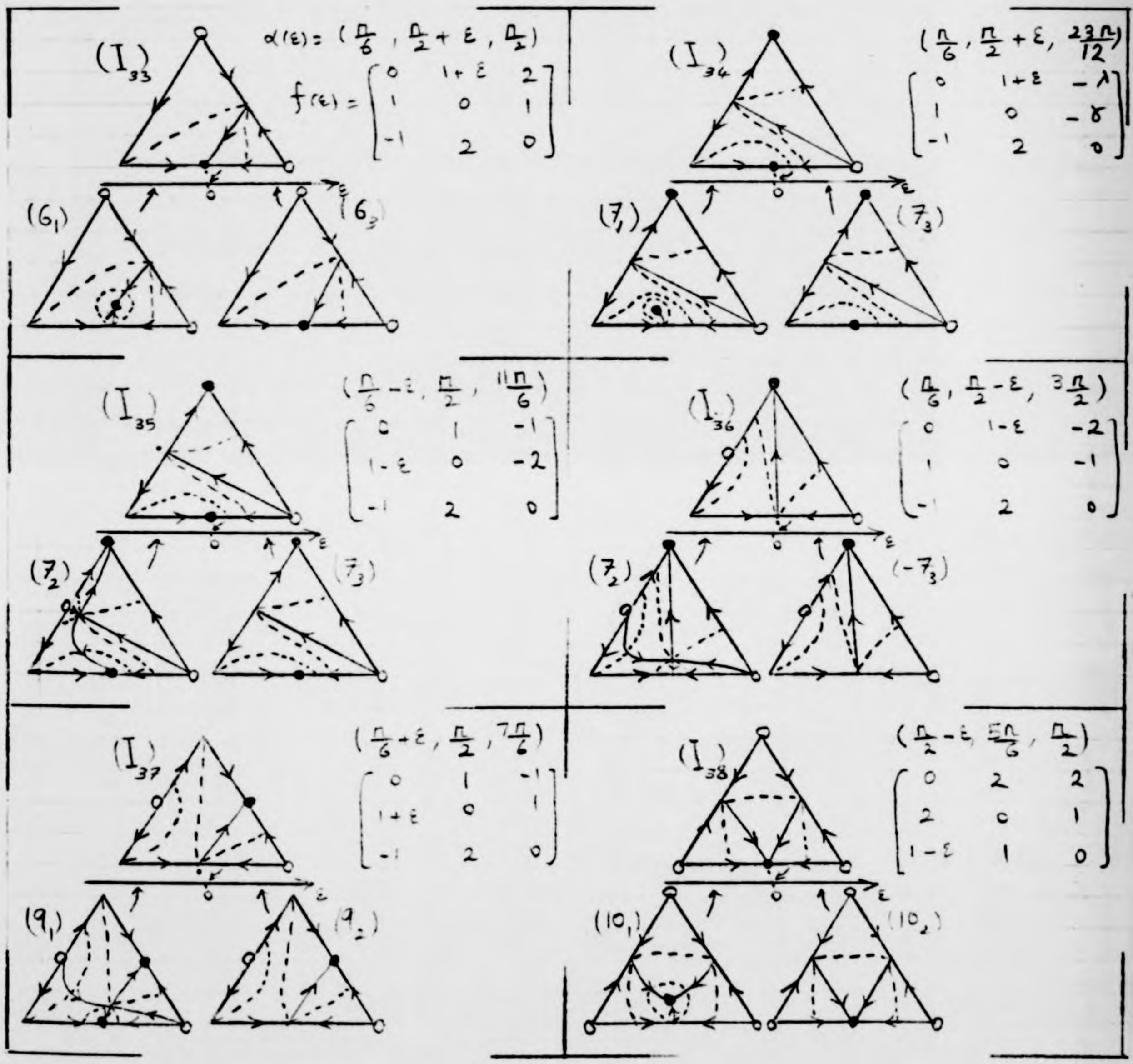
(I_{32})



(6_2) (6_4)

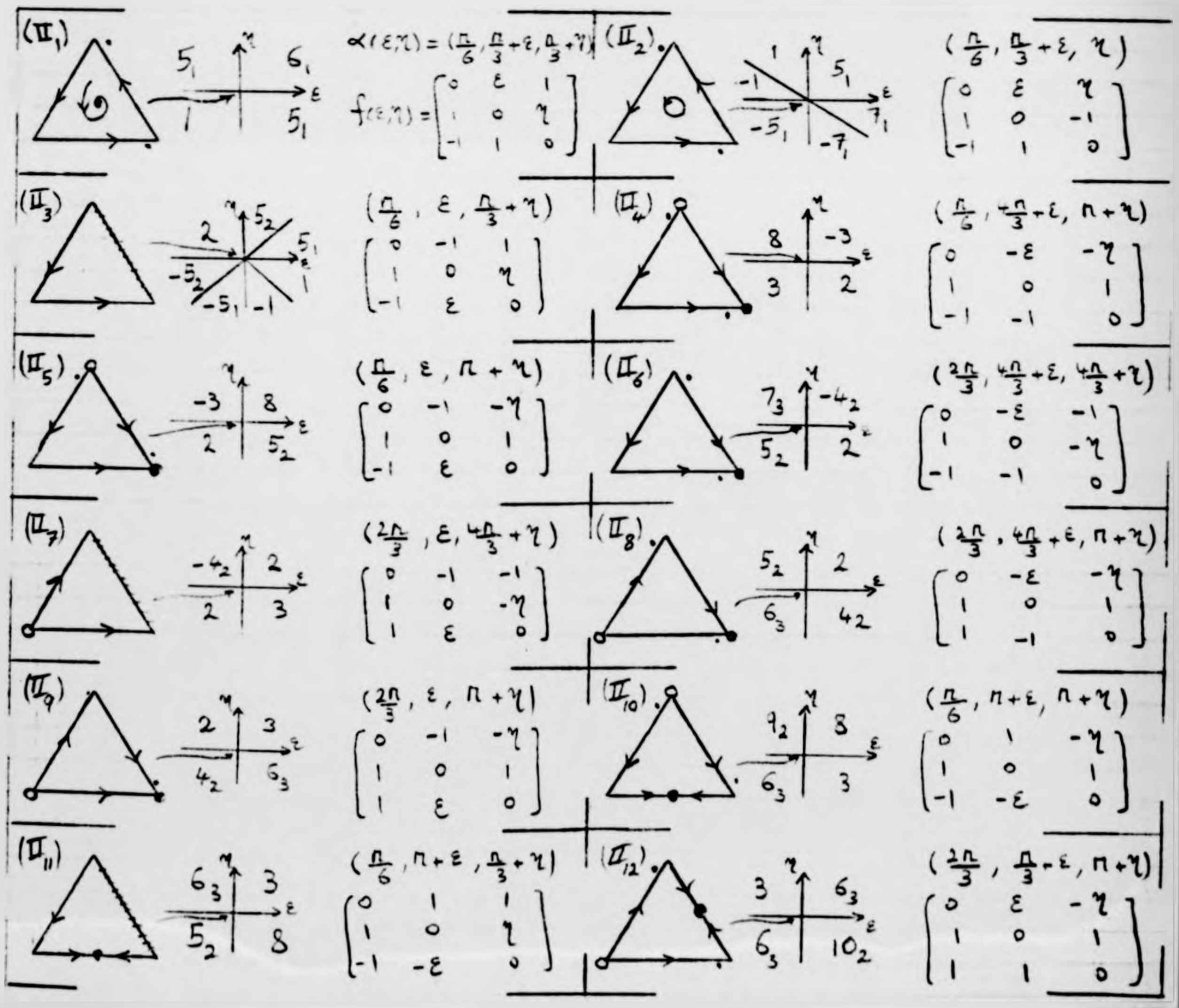
$$\left(\frac{\pi}{6}, \frac{\pi}{2} - \varepsilon, \frac{11\pi}{12}\right)$$

$$\begin{bmatrix} 0 & 1-\varepsilon & \lambda \\ 1 & 0 & \gamma \\ -1 & 2 & 0 \end{bmatrix}$$


(I33)
α(ε) = (π/6, π/2 + ε, π/2)
f(ε) = [0 1+ε 2; 1 0 1; -1 2 0]
(6₁) (6₃)
(I34)
(π/6, π/2 + ε, 23π/12)
[0 1+ε -λ; 1 0 -δ; -1 2 0]
(7₁) (7₃)
(I35)
(π/6 - ε, π/2, 11π/6)
[0 1 -1; 1-ε 0 -2; -1 2 0]
(7₂) (7₃)
(I36)
(π/6, π/2 - ε, 3π/2)
[0 1-ε -2; 1 0 -1; -1 2 0]
(7₂) (-7₃)
(I37)
(π/6 + ε, π/2, 7π/6)
[0 1 -1; 1+ε 0 1; -1 2 0]
(9₁) (9₂)
(I38)
(π/2 - ε, 5π/6, π/2)
[0 2 2; 2 0 1; 1-ε 1 0]
(10₁) (10₂)
ε
0

Appendix 2

We present here an example of a matrix in each of the cod 2 strata Π_i $(1 \le i \le 53)$ together with a transversal deformation of that matrix. The nearby stable matrices are indicated by their labels. A dashed line implies that the line is pointwise fixed.

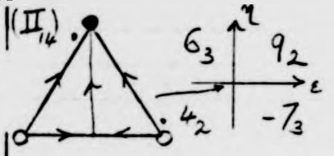


(II_{13})

$\alpha(\varepsilon,\eta) = (\frac{\pi}{2}, \pi+\varepsilon, \eta)$

$f(\varepsilon,\eta) = \begin{bmatrix} 0 & 1 & \eta \\ 2 & 0 & -1 \\ 1 & -\varepsilon & 0 \end{bmatrix}$

(II_{14})



$(\frac{5\pi}{6}, \pi+\varepsilon, \eta)$

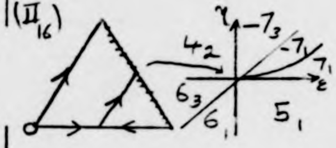
$\begin{bmatrix} 0 & 1 & \eta \\ 1 & 0 & -1 \\ 2 & -\varepsilon & 0 \end{bmatrix}$

(II_{15})

$(\frac{\pi}{2}, \pi+\varepsilon, \frac{4\pi}{3}+\eta)$

$\begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & -\eta \\ 1 & -\varepsilon & 0 \end{bmatrix}$

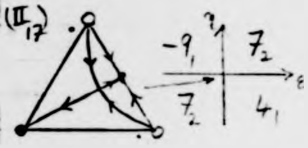
(II_{16})



$(\frac{5\pi}{6}, \pi+\varepsilon, \frac{4\pi}{3}+\eta)$

$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -\eta \\ 2 & -\varepsilon & 0 \end{bmatrix}$

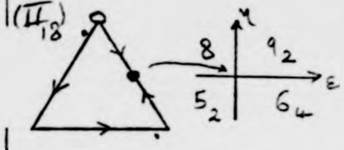
(II_{17})



$(\frac{5\pi}{3}, \frac{\pi}{3}+\varepsilon, \pi+\eta)$

$\begin{bmatrix} 0 & \varepsilon & -\eta \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

(II_{18})



$(\frac{\pi}{6}, \frac{\pi}{3}+\varepsilon, \pi+\eta)$

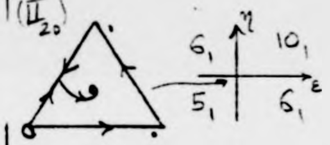
$\begin{bmatrix} 0 & \varepsilon & -\eta \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

(II_{19})

$(\frac{\pi}{6}, \pi+\varepsilon, \eta)$

$\begin{bmatrix} 0 & 1 & -\eta \\ 1 & 0 & -1 \\ -1 & -\varepsilon & 0 \end{bmatrix}$

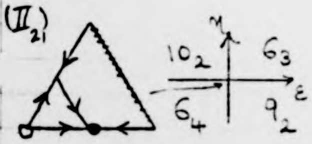
(II_{20})



$(\frac{\pi}{2}, \frac{\pi}{3}+\varepsilon, \pi+\eta)$

$\begin{bmatrix} 0 & \varepsilon & -\eta \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

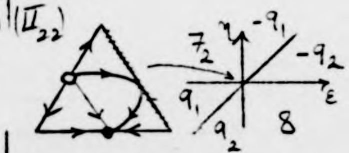
(II_{21})



$(\frac{\pi}{2}, \pi+\varepsilon, \frac{\pi}{3}+\eta)$

$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & \eta \\ 1 & -\varepsilon & 0 \end{bmatrix}$

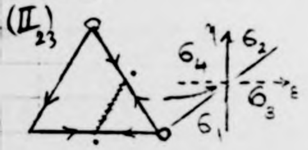
(II_{22})



$(\frac{\pi}{6}, \pi+\varepsilon, \frac{4\pi}{3}+\eta)$

$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -\eta \\ -1 & -\varepsilon & 0 \end{bmatrix}$

(II_{23})



$(\frac{\pi}{6}, \frac{\pi}{2}+\varepsilon, \frac{5\pi}{6}+\eta)$

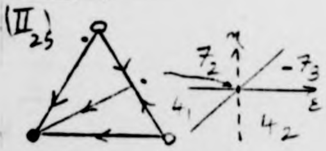
$\begin{bmatrix} 0 & 1+\varepsilon & 1-\eta \\ 1 & 0 & 2 \\ -1 & 2 & 0 \end{bmatrix}$

(II_{24})

$(\frac{\pi}{6}, \frac{\pi}{2}+\varepsilon, \frac{11\pi}{6}+\eta)$

$\begin{bmatrix} 0 & 1+\varepsilon & -1+\eta \\ 1 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix}$

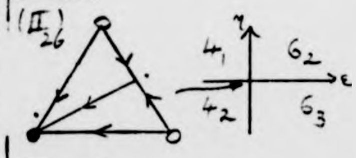
(II_{25})



$(\frac{5\pi}{3}, \frac{2\pi}{3}+\varepsilon, \pi+\eta)$

$\begin{bmatrix} 0 & 1 & -\eta \\ -1 & 0 & 1 \\ -1 & 1+\varepsilon & 0 \end{bmatrix}$

(II_{26})



$(\frac{4\pi}{3}+\varepsilon, \frac{\pi}{2}, \frac{5\pi}{6}+\eta)$

$\begin{bmatrix} 0 & 1 & 1-\eta \\ -1 & 0 & 2 \\ -\varepsilon & 2 & 0 \end{bmatrix}$

(II_{27})

$(\frac{\pi}{6}, \frac{\pi}{3}+\varepsilon, \frac{2\pi}{3}+\eta)$

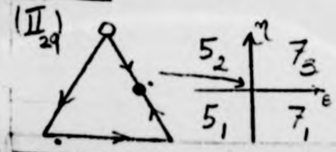
$\begin{bmatrix} 0 & \varepsilon & 1-\eta \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

(II_{28})

$(\frac{\pi}{3}+\varepsilon, \frac{\pi}{6}, \frac{\pi}{2}+\eta)$

$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1+\eta \\ \varepsilon & 1 & 0 \end{bmatrix}$

(II_{29})



$(\varepsilon, \frac{\pi}{6}, \frac{\pi}{2}+\eta)$

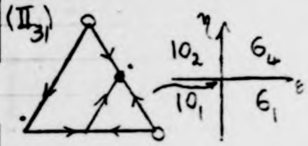
$\begin{bmatrix} 0 & -1 & 2 \\ \varepsilon & 0 & 1+\eta \\ -1 & 1 & 0 \end{bmatrix}$

(II_{30})

$(\frac{\pi}{6}, \frac{2\pi}{3}+\varepsilon, \pi+\eta)$

$\begin{bmatrix} 0 & 1+\varepsilon & -\eta \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

(II_{31})



$$\alpha(\varepsilon,\eta)=\left(\frac{\pi}{3}-\varepsilon,\ \frac{\pi}{2},\ \frac{5\pi}{6}+\eta\right)$$

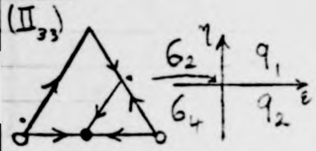
$$f(\varepsilon,\eta)=\begin{bmatrix}0 & 1 & 1-\eta\\ 1 & 0 & 2\\ -\varepsilon & 2 & 0\end{bmatrix}$$

(II_{32})

$$\left(\frac{2\pi}{3},\ \frac{2\pi}{3}+\varepsilon,\ \pi+\eta\right)$$

$$\begin{bmatrix}0 & 1+\varepsilon & -\eta\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$$

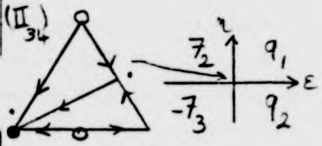
(II_{33})



$$\left(\frac{\pi}{3}+\varepsilon,\ \frac{5\pi}{6},\ \frac{7\pi}{6}+\eta\right)$$

$$\begin{bmatrix}0 & 2 & -1\\ 1 & 0 & 1-\eta\\ \varepsilon & 1 & 0\end{bmatrix}$$

(II_{34})



$$\left(\frac{4\pi}{3}+\varepsilon,\ \frac{\pi}{6},\ \frac{\pi}{2}+\eta\right)$$

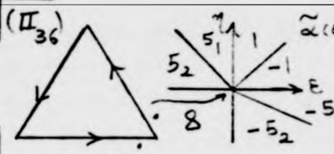
$$\begin{bmatrix}0 & -1 & 2\\ -1 & 0 & 1+\eta\\ -\varepsilon & 1 & 0\end{bmatrix}$$

(II_{35})

$$\left(\frac{5\pi}{6},\ \frac{\pi}{3}-\varepsilon,\ \frac{\pi}{3}+\eta\right)$$

$$\begin{bmatrix}0 & \varepsilon & 1\\ 1 & 0 & \eta\\ 2 & 1 & 0\end{bmatrix}$$

(II_{36})



$$\tilde{\alpha}(\varepsilon,\eta)=\left(\frac{\pi}{6},\ \varepsilon,\ \eta,\ \frac{\pi}{6}\right)$$

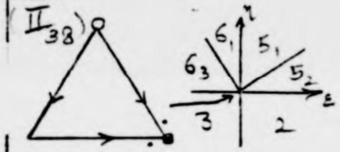
$$\begin{bmatrix}0 & -\varepsilon & 1\\ 1 & 0 & -1\\ -1 & \eta & 0\end{bmatrix}$$

(II_{37})

$$\left(\frac{\pi}{6},\ \varepsilon,\ \eta,\ \frac{7\pi}{6}\right)$$

$$\begin{bmatrix}0 & -\varepsilon & -1\\ 1 & 0 & 1\\ -1 & \eta & 0\end{bmatrix}$$

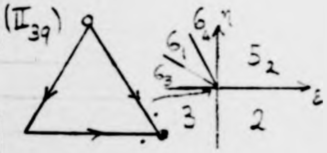
(II_{38})



$$\left(\frac{\pi}{6},\ \varepsilon,\ \eta,\ \frac{\pi}{2}\right)$$

$$\begin{bmatrix}0 & -\varepsilon & 2\\ 1 & 0 & 1\\ -1 & \eta & 0\end{bmatrix}$$

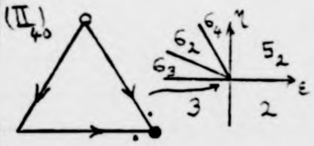
(II_{39})



$$\left(\frac{\pi}{6},\ \varepsilon,\ \eta,\ \frac{9\pi}{12}\right)$$

$$\begin{bmatrix}0 & -\varepsilon & \sqrt{2}\\ 1 & 0 & \gamma\\ -1 & \eta & 0\end{bmatrix}$$

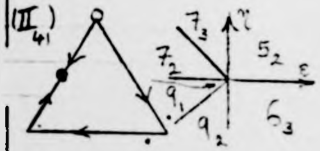
(II_{40})



$$\left(\frac{\pi}{6},\ \varepsilon,\ \eta,\ \frac{11\pi}{12}\right)$$

$$\begin{bmatrix}0 & -\varepsilon & \lambda\\ 1 & 0 & \gamma\\ -1 & \eta & 0\end{bmatrix}$$

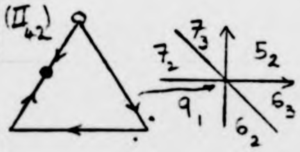
(II_{41})



$$\left(\frac{\pi}{6},\ \varepsilon,\ \eta,\ \frac{3\pi}{2}\right)$$

$$\begin{bmatrix}0 & -\varepsilon & -2\\ 1 & 0 & -1\\ -1 & \eta & 0\end{bmatrix}$$

(II_{42})



$$\left(\frac{\pi}{6},\ \varepsilon,\ \eta,\ \frac{7}{4}\pi\right)$$

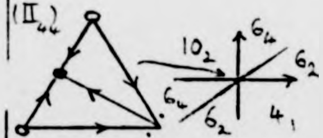
$$\begin{bmatrix}0 & -\varepsilon & -\sqrt{2}\\ 1 & 0 & -\gamma\\ -1 & \eta & 0\end{bmatrix}$$

(II_{43})

$$\left(\frac{\pi}{6},\ \varepsilon,\ \eta,\ \frac{23}{12}\pi\right)$$

$$\begin{bmatrix}0 & -\varepsilon & -\lambda\\ 1 & 0 & \gamma\\ -1 & \eta & 0\end{bmatrix}$$

(II_{44})



$$\left(\frac{5\pi}{6},\ \varepsilon,\ \eta,\ \frac{5\pi}{12}\right)$$

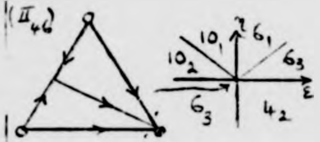
$$\begin{bmatrix}0 & -\varepsilon & \gamma\\ 1 & 0 & \lambda\\ 2 & \eta & 0\end{bmatrix}$$

(II_{45})

$$\left(\frac{\pi}{2},\ \varepsilon,\ \eta,\ \frac{5\pi}{6}\right)$$

$$\begin{bmatrix}0 & -\varepsilon & 1\\ 2 & 0 & 2\\ 1 & \eta & 0\end{bmatrix}$$

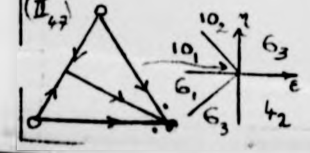
(II_{46})



$$\left(\frac{\pi}{2},\ \varepsilon,\ \eta,\ \frac{\pi}{2}\right)$$

$$\begin{bmatrix}0 & -\varepsilon & 2\\ 2 & 0 & 1\\ 1 & \eta & 0\end{bmatrix}$$

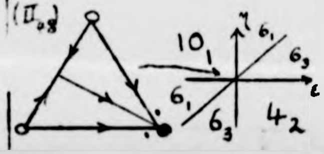
(II_{47})



$$\left(\frac{5\pi}{6},\ \varepsilon,\ \eta,\ \frac{5\pi}{6}\right)$$

$$\begin{bmatrix}0 & -\varepsilon & 1\\ 1 & 0 & 2\\ 2 & \eta & 0\end{bmatrix}$$

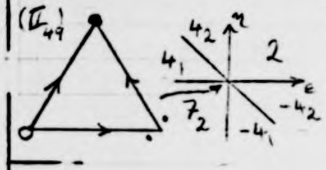
(II_{48})



$$\left(\frac{5\pi}{6},\ \varepsilon,\ \eta,\ \frac{7\pi}{6}\right)$$

$$\begin{bmatrix}0 & -\varepsilon & \gamma\\ 1 & 0 & \sqrt{2}\\ 2 & \eta & 0\end{bmatrix}$$

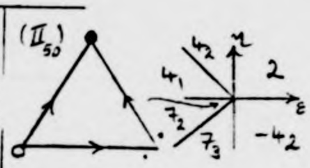
(II_{49})



$\tilde{\alpha}(\varepsilon,\eta) = (\frac{\pi}{2}, \varepsilon, \eta, \frac{11\pi}{6})$

$$f(\varepsilon,\eta) = \begin{bmatrix} 0 & -\varepsilon & -1 \\ 2 & 0 & -2 \\ 1 & \eta & 0 \end{bmatrix}$$

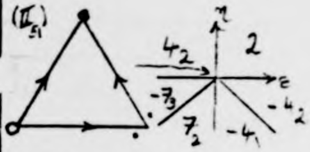
(II_{50})



$(\frac{\pi}{2}, \varepsilon, \eta, \frac{3\pi}{2})$

$$\begin{bmatrix} 0 & -\varepsilon & -2 \\ 2 & 0 & -1 \\ 1 & \eta & 0 \end{bmatrix}$$

(II_{51})



$(\frac{5\pi}{6}, \varepsilon, \eta, \frac{11\pi}{6})$

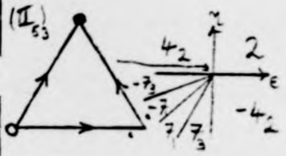
$$\begin{bmatrix} 0 & -\varepsilon & -1 \\ 1 & 0 & -2 \\ 2 & \eta & 0 \end{bmatrix}$$

(II_{52})

$(\frac{5\pi}{6}, \varepsilon, \eta, \frac{19\pi}{12})$

$$\begin{bmatrix} 0 & -\varepsilon & -\gamma \\ 1 & 0 & -\sqrt{2} \\ 2 & \eta & 0 \end{bmatrix}$$

(II_{53})



$(\frac{11\pi}{12}, \varepsilon, \eta, \frac{5\pi}{4})$

$$\begin{bmatrix} 0 & -\varepsilon & -\gamma \\ \lambda & 0 & -\lambda \\ \gamma & \eta & 0 \end{bmatrix}$$

Appendix 3

(a) We will prove our claim, in proposition 6.1 (ii), that the discriminant of the quadratic $\bar{x}_1^2[(b_1+b_3)-(b_1-b_3)^2]$ + $\bar{x}_2[(b_2+b_3)-(b_2-b_3)^2] + 2\bar{x}_1\bar{x}_2[2(b_1+b_2)-2(b_1+b_2)^2-b_1b_2]$ is given by $\Delta = -36\ b_1b_2b_3 < 0$. Let $S = b_1+b_2$ and $p = b_1b_2$. We express Δ in terms of p and s :

$$\frac{\Delta}{4} = [2(b_1+b_2)-2(b_1+b_2)^2-b_1b_2]^2-[(b_1+b_3)-(b_1-b_3)^2][(b_2+b_3)-(b_2-b_3)^2] =$$
$$(2s-2s^2-p)^2-\{(1-b_1)(1-b_2)+[b_1b_2-(1-b_1-b_2)(b_1+b_2)+(1-b_1-b_2)^2]^2$$
$$-(1-b_1)(2b_1+b_2-1)^2-(1-b_2)(2b_2+b_1-1)^2\} = (2s-2s^2-p)^2-\{(1+p-s)$$
$$+[p-(1-s)s+(1-s)^2]^2-(1-b_1)(s+b_1-1)^2-(1-b_2)(s+b_2-1)^2\} =$$
$$(2s-2s^2-p)^2-\{(1+p-s)+[p+(1-s)(1-2s)]^2-[s^2(1-b_1)+s^2(1-b_2)]$$
$$-[(1-b_1)^3+(1-b_2)^3] + 2s(s^2-2p-2s+2)\} = [2s(1-s)-p]^2-(1+p-s)$$
$$-[p+(1-s)(1-2s)]^2 + 2s^2-s^3 +[2-(b_1^3+b_2^3)-3(b_1+b_2)+3(b_1^2+b_2^2)]$$
$$-2s^3 + 4sp + 4s^2-4s = [2s(1-s)-p]^2 - [2s(1-s)-p-(1-s)]^2$$
$$+ 2s^2-s^3-2s^3 + 4sp +4s^2-4s + [2-s(s^2-3p)-3s+3s^2-6p]$$
$$-(1+p-s) = -(1-s)^2 + 4s(1-s)^2 - 2p(1-s)-3s^3 + 6s^2 + 4sp$$
$$-4s + 2-s^3 + 3sp-3s + 3s^2 - 6p -(1+p-s) = -(1+s^2-2s)$$
$$+ 4s + 4s^3 - 8s^2-2p + 2ps-3s^3 + 6s^2 + 4sp - 4s + 2-s^3 + 3sp$$
$$-3s + 3s^2 - 6p -1 -p+s = 9ps-9p = 9p(s-1) = 9b_1b_2(b_1+b_2-1)$$
$$= -9b_1b_2b_3 \quad .$$

Therefore $\Delta = -36\ b_1b_2b_3 < 0$ as claimed.

(b) We will calculate the expression for $\ell(\theta_1,\theta_2,x)$ in proposition 6.9 (ii):

$$\ell(\theta_1,\theta_2,x) = V_1^{\theta_1}(x)V_2^{\theta_2}(x) - V_1^{\theta_2}(x)V_2^{\theta_1}(x) = (\theta_1-\theta_2)x_1x_2\{(x_2+x_3-2x_1x_2-2x_2x_3$$

$$-2x_3x_1)[a_2x_3-a_2x_1-(a_1-a_2)x_1x_2-(a_2-a_3)x_2x_3-(a_3-a_1)x_3x_1]-(x_3+x_1-2x_1x_2$$

$$-2x_2x_3-2x_3x_1)[a_1x_2-a_1x_3-(a_1-a_2)x_1x_2-(a_2-a_3)x_2x_3-(a_3-a_1)x_3x_1] .$$

Substituting $x_3 = 1-x_1-x_2$ this gives:

$$\frac{\ell(\theta_1,\theta,x)}{(\theta_1-\theta_2)x_1x_2} = [1-x_1-2x_1x_2-2(x_1+x_2)(1-x_1-x_2)][a_2(1-x_1-x_2)$$

$$-a_2x_1-(a_1-a_2)x_1x_2-(a_2-a_3)x_2(1-x_1-x_2)-(a_3-a_1)x_1(1-x_1-x_2)]$$

$$-[1-x_2-2x_1x_2-2(x_1+x_2)(1-x_1-x_2)][a_1x_2-a_1(1-x_1-x_2)$$

$$-(a_1-a_2)x_1x_2-(a_2-a_3)x_2(1-x_1-x_2)-(a_3-a_1)x_1(1-x_1-x_2)] =$$

$$[1-x_1-2x_1x_2-2x_1-2x_2+2x_1^2+2x_2^2+4x_1x_2][a_2-a_2x_1-a_2x_2$$

$$-a_2x_1-a_1x_1x_2+a_2x_1x_2-(a_2-a_3)x_2+(a_2-a_3)x_1x_2+(a_2-a_3)x_2^2$$

$$-(a_3-a_1)x_1+(a_3-a_1)x_1^2+(a_3-a_1)x_1x_2] - [1-x_2-2x_1x_2-2x_1-2x_2$$

$$+2x_1^2+2x_2^2+4x_1x_2][a_1x_2-a_1+a_1x_1+a_1x_2-(a_1-a_2)x_1x_2$$

$$-(a_2-a_3)x_2+(a_2-a_3)x_1x_2+(a_2-a_3)x_2^2-(a_3-a_1)x_1+(a_3-a_1)x_1^2$$

$$+(a_3-a_1)x_1x_2] = [1-3x_1-2x_2+2x_1^2+2x_2^2+2x_1x_2]\ [a_2$$

$$+(a_1-2a_2-a_3)x_1 + (a_3-2a_2)x_2+(-2a_1+2a_2)x_1x_2 + (a_2-a_3)x_2^2$$

$$+(a_3-a_1)x_1^2] - [1-2x_1-3x_2+2x_1^2+2x_2^2+2x_1x_2]\ [-a_1$$

$$+(2a_1-a_3)x_1 + (2a_1-a_2+a_3)x_2 + (-2a_1+2a_2)x_1x_2+(a_2-a_3)x_2^2$$

$$+(a_3-a_1)x_1^2] = a_2+(a_1-2a_2-a_3)x_1 + (a_3-2a_2)x_2+(-2a_1+2a_2)x_1x_2$$

$$
\begin{aligned}
&+(a_2-a_3)x_2^2+(a_3-a_1)x_1^2-3a_2x_1-3(a_1-2a_2-a_3)x_1^2-3(a_3-2a_2)x_1x_2\\
&-6(-a_1+a_2)x_1^2x_2-3(a_2-a_3)x_1x_2^2-3(a_3-a_1)x_1^3-2a_2x_2\\
&-2(a_1-2a_2-a_3)x_1x_2-2(a_3-2a_2)x_2^2-4(-a_1+a_2)x_1x_2^2\\
&-2(a_2-a_3)x_2^3-2(a_3-a_1)x_1^2x_2+2a_2x_1^2+2(a_1-2a_2-a_3)x_1^3\\
&+2(a_3-2a_2)x_1^2x_2+2a_2x_2^2+2(a_1-2a_2-a_3)x_1^2x_2+2(a_3-2a_2)x_2^3\\
&+2a_2x_1x_2+2(a_1-2a_2-a_3)x_1^2x_2+2(a_3-2a_2)x_1x_2^2+a_1-(2a_1-a_3)x_1-(2a_1-a_2+a_3)x_2\\
&-(-2a_1+2a_2)x_1x_2-(a_2-a_3)x_2^2-(a_3-a_1)x_1^2-2a_1x_1+2(2a_1-a_3)x_1^2\\
&+2(2a_1-a_2+a_3)x_1x_2+4(-a_1+a_2)x_1^2x_2+2(a_2-a_3)x_1x_2^2\\
&+2(a_3-a_1)x_1^3-3a_1x_2+3(2a_1-a_3)x_1x_2+3(2a_1-a_2+a_3)x_2^2\\
&+6(-a_1+a_2)x_1x_2^2+3(a_2-a_3)x_2^3+3(a_3-a_1)x_1^2x_2+2a_1x_1^2\\
&-2(2a_1-a_3)x_1^3-2(2a_1-a_2+a_3)x_1^2x_2+2a_1x_2^2\\
&-2(2a_1-a_3)x_1x_2^2-2(2a_1-a_2+a_3)x_2^3+2a_1x_1x_2\\
&-2(2a_1-a_3)x_1^2x_2-2(2a_1-a_2+a_3)x_1x_2^2 = (a_1+a_2)\\
&-(3a_1+5a_2)x_1-(5a_1+3a_2)x_2+(3a_1+8a_2+a_3)x_1^2+(8a_1+3a_2+a_3)x_2^2\\
&+2(5a_1+5a_2-a_3)x_1x_2-(a_1+4a_2+a_3)x_1^3-(4a_1+a_2+a_3)x_2^3\\
&+(-5a_1-8a_2+a_3)x_1^2x_2+(-8a_1-5a_2+a_3)x_1x_2^2
\end{aligned}
$$

as claimed in the proposition.

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